

A generalization for a finite family of functions of the converse of Browder's fixed point theorem

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Abstract. Taking as model the attractor of an iterated function system consisting of φ -contractions on a complete and bounded metric space, we introduce the set-theoretic concept of family of functions having attractor. We prove that, given such a family, there exist a metric on the set on which the functions are defined and take values and a comparison function φ such that all the family's functions are φ -contractions. In this way we obtain a generalization for a finite family of functions of the converse of Browder's fixed point theorem. As byproducts we get a particular case of Bessaga's theorem concerning the converse of the contraction principle and a companion of Wong's result which extends the above mentioned Bessaga's result for a finite family of commuting functions with common fixed point.

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1. INTRODUCTION

The problem of the converse of Banach-Picard-Caccioppoli principle was treated by several mathematicians each of them concentrating on different assumptions. C. Bessaga (see [3], [10] and [13]) was the first one to treat the problem by using only set-theoretic assumptions. J. S. W. Wong (see [24]) extended Bessaga's result for a finite family of commuting functions with common unique fixed point. Other results on this direction are due to L. Janoš (see [12]), P.R. Meyers (see [18]) and S. Leader (see [15]).

The idea of replacing the contractivity condition imposed on the function $f : X \rightarrow X$ considered in the Banach-Picard-Caccioppoli principle by a

weaker one described by the inequality $d(f(x), f(y)) \leq \varphi(d(x, y))$ for all $x, y \in X$, where φ has certain properties defining the so called comparison function, was treated, among others, by D.W. Boyd and J.S. Wong (see [4]), F. Browder (see [5]), J. Matkowski (see [17]) and I. A. Rus (see [21]). A function f satisfying the previous inequality is called φ -contractions. From the point of view of the problem treated in this paper a special place is played by Browder's result concerning φ -contractions (see Theorem 2.5). For more details about this result one can consult [11].

Iterated function systems, introduced by J. Hutchinson (see [9]) and popularized by M. Barnsley (see [1]), represent one of the most general way to generate fractals. The large variety of their applications is the background of the current effort to extend the classical Hutchinson's theory. One line of research in this direction is to weaken the usual contraction condition by considering iterated function systems consisting of φ -contractions. For results in this direction one can consult [7], [8], [22] and [23].

By selecting some properties of the attractor of an iterated function system consisting of φ -contractions on a complete and bounded metric space, we introduced the set-theoretic concept of family of functions having attractor (Definition 3.3). We prove that, given such a family, there exist a complete and bounded metric on the set on which the functions are defined and take values and a comparison function φ such that all the family's functions are φ -contractions (see Theorem 3.21). In this way we obtain a generalization for a finite family of functions of the converse of Browder's fixed point theorem.

If $\mathcal{F} = (f_i)_{i \in I}$ is a family of functions having attractor A , where $f_i : X \rightarrow X$ and I is finite, we obtain the result tracking the following steps:

- the construction (based on the main result from [19]) of a metric d on A and a comparison function φ such that $d(f_i(x), f_i(y)) \leq \varphi(d(x, y))$ for every $i \in I$ and every $x, y \in A$, i.e. f_i 's are φ -contractions on the attractor with respect to d (Theorem 3.4)
- the construction of a semi-metric d^μ on X , associated to \mathcal{F} and to a sequence μ , such that $d^\mu(f_i(x), f_i(y)) \leq d^\mu(x, y)$ for every $x, y \in X$, i.e. f_i 's are nonexpansive on X with respect to d^μ (Proposition 3.8)
- the construction of a complete and bounded metric d on X (Proposition 3.16)
- the construction of a comparison function φ such that $d(f_i(x), f_i(y)) \leq \varphi(d(x, y))$ for every $i \in I$ and every $x, y \in X$, i.e. f_i 's are φ -contractions with respect to d (Lemma 3.20).

Finally we present a result which removes the boundedness condition on

the metric d , we point out that one can obtain from our result a particular case of Bessaga's theorem concerning the converse of the contraction principle (see Theorem 5 from [10]) and we present a companion of Wong's result which extends the above mentioned Bessaga's result for a finite family of commuting functions with common fixed point (see [24]).

2. PRELIMINARIES

For a function $f : X \rightarrow X$ and $n \in \mathbb{N}$, by $f^{[n]}$ we mean the composition of f by itself n times.

Definition 2.1 (comparison function). *A function $\varphi : [0, \infty) \rightarrow [0, \infty)$ is called a comparison function if it has the following three properties:*

- i) φ is increasing;
- ii) $\varphi(t) < t$ for every $t > 0$;
- iii) φ is right-continuous.

Remark 2.2.

i) Any function $\varphi : [0, \infty) \rightarrow [0, \infty)$ satisfying ii) and iii) from the above definition has the following property: $\lim_{n \rightarrow \infty} \varphi^{[n]}(t) = 0$ for every $t > 0$ (see Remark 1 from [16]).

ii) $\varphi(0) = 0$ for every comparison function.

Definition 2.3 (φ -contraction). *Let (X, d) be a metric space and a function $\varphi : [0, \infty) \rightarrow [0, \infty)$. A function $f : X \rightarrow X$ is called a φ -contraction if $d(f(x), f(y)) \leq \varphi(d(x, y))$ for all $x, y \in X$.*

Remark 2.4. Every φ -contraction is Lipschitz, so it is continuous.

The next result is known as Browder's Theorem.

Theorem 2.5 (see Theorem 1 from [5], Theorem 1 from [11] or Example 2.9., 1) from [2]). *Let (X, d) be a complete and bounded metric space and $\varphi : [0, \infty) \rightarrow [0, \infty)$ a comparison function. Then every φ -contraction $f : X \rightarrow X$ has a unique fixed point x_0 and $\lim_{n \rightarrow \infty} f^{[n]}(x) = x_0$ for every $x \in X$.*

Given a metric space (X, d) and a subset Y of X , by $d(Y)$ we denote the diameter of Y and by $\mathcal{K}(X)$ we denote the family of non-empty compact subsets of X .

For a nonempty set I , by $\Lambda(I)$ we mean the set $I^{\mathbb{N}^*}$ and by $\Lambda_n(I)$ we mean the set $I^{\{1,2,\dots,n\}}$. So, the elements of $\Lambda(I)$ are written as infinite words $\alpha = \alpha_1\alpha_2\dots\alpha_m\alpha_{m+1}\dots$ and the elements of $\Lambda_n(I)$ are written as finite words $\alpha = \alpha_1\alpha_2\dots\alpha_n$ (n , which is the length of ω , is denoted by $|\omega|$).

By $\Lambda^*(I)$ we denote the set of all finite words, i.e. $\Lambda^*(I) \stackrel{def}{=} \bigcup_{n \in \mathbb{N}^*} \Lambda_n(I) \cup \{\lambda\}$, where λ is the empty word.

For $\alpha = \alpha_1\alpha_2\dots\alpha_m\alpha_{m+1}\dots \in \Lambda(I)$ and $n \in \mathbb{N}$, we shall use the following notation: $[\alpha]_n \stackrel{not}{=} \alpha_1\alpha_2\dots\alpha_n$ if $n \geq 1$ and λ if $n = 0$.

For two words $\alpha \in \Lambda_n(B)$ and $\beta \in \Lambda_m(B)$ or $\beta \in \Lambda(B)$, by $\alpha\beta$ we mean the concatenation of the words α and β , i.e. $\alpha\beta = \alpha_1\alpha_2\dots\alpha_n\beta_1\beta_2\dots\beta_m$ and respectively $\alpha\beta = \alpha_1\alpha_2\dots\alpha_n\beta_1\beta_2\dots\beta_m\beta_{m+1}\dots$.

On $\Lambda(I)$ we consider the metric given by $d_\Lambda(\alpha, \beta) = \sum_{k=1}^{\infty} \frac{1-\delta_{\alpha_k}^{\beta_k}}{3^k}$, where

$$\delta_x^y = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}.$$

Remark 2.6. The function $\tau_i : \Lambda(I) \rightarrow \Lambda(I)$, given by $\tau_i(\alpha) = i\alpha$ for every $\alpha \in \Lambda(I)$, is continuous.

Remark 2.7.

i) The convergence in the compact metric space $(\Lambda(I), d_\Lambda)$ is the convergence on components.

ii) If I is finite, then $(\Lambda(I), d_\Lambda)$ is compact.

Given the functions $f_i : X \rightarrow X$, where X is a given set and $i \in I$, we shall use the following notations:

i) $f_\lambda = Id_X$;

ii) $f_{\alpha_1\alpha_2\dots\alpha_m} \stackrel{not}{=} f_{\alpha_1} \circ f_{\alpha_2} \circ \dots \circ f_{\alpha_m}$ for every $\alpha_1, \alpha_2, \dots, \alpha_m \in I$;

iii) $Y_\alpha \stackrel{not}{=} f_\alpha(Y)$ for every $\alpha \in \Lambda^*(I)$ and every $Y \subseteq X$.

Definition 2.8 (topological self-similar set, topological self-similar system). *A compact Hausdorff topological space K is called a topological self-similar set if there exist continuous functions $f_1, f_2, \dots, f_N : K \rightarrow K$, where $N \in \mathbb{N}^*$, and a continuous surjection $\pi : \Lambda(\{1, 2, \dots, N\}) \rightarrow K$ such that the diagram*

$$\begin{array}{ccc} \Lambda(\{1, 2, \dots, N\}) & \xrightarrow{\tau_i} & \Lambda(\{1, 2, \dots, N\}) \\ \pi \downarrow & & \downarrow \pi \\ K & \xrightarrow{f_i} & K \end{array}$$

commutes for all $i \in \{1, 2, \dots, N\}$.

We say that $(K, (f_i)_{i \in \{1, 2, \dots, N\}})$, a topological self-similar set together with the set of continuous maps as above, is a topological self-similar system.

The above definition is Definition 0.3 from [14].

Theorem 2.9 (see Theorem 3.1 from [19]). *For every topological self-similar system $(K, (f_i)_{i \in \{1, 2, \dots, N\}})$ there exist a metric δ on K which is compatible with the original topology and a comparison function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that $\delta(f_i(x), f_i(y)) \leq \varphi(\delta(x, y))$ for each $i \in \{1, 2, \dots, N\}$ and each $x, y \in K$.*

Definition 2.10 (iterated function system). *Given a complete metric space (X, d) , an iterated function system is a pair $\mathcal{S} = ((X, d), (f_i)_{i \in \{1, 2, \dots, N\}})$, where $f_i : X \rightarrow X$ is a continuous function for each $i \in \{1, 2, \dots, N\}$, $N \in \mathbb{N}^*$.*

3. THE RESULTS

Some considerations on iterated function systems consisting of φ -contractions

We start with a result that emphasizes some properties of iterated function systems consisting of φ -contractions.

Proposition 3.1. *Let us consider an iterated function system $\mathcal{S} = ((X, d), (f_i)_{i \in I})$ consisting of φ -contractions, where φ is a comparison function and the metric space (X, d) is complete and bounded. Then:*

a) *For every $\alpha \in \Lambda(I)$, the set $\bigcap_{n \in \mathbb{N}^*} X_{[\alpha]_n}$ has a unique element which is denoted by a_α .*

b) *If $a_\alpha \neq a_\beta$, where $\alpha, \beta \in \Lambda(I)$, then there exists $n_0 \in \mathbb{N}^*$ such that $X_{[\alpha]_{n_0}} \cap X_{[\beta]_{n_0}} = \emptyset$.*

Proof.

a) Let us consider $\alpha = \alpha_1 \alpha_2 \dots \alpha_m \dots \in \Lambda(I)$ and $n \in \mathbb{N}^*$. As for every $x, y \in X_{[\alpha]_n}$ there exist $u, v \in X$ such that $x = f_{\alpha_1 \alpha_2 \dots \alpha_n}(u)$ and $y = f_{\alpha_1 \alpha_2 \dots \alpha_n}(v)$, we have $d(x, y) = d(f_{\alpha_1 \alpha_2 \dots \alpha_n}(u), f_{\alpha_1 \alpha_2 \dots \alpha_n}(v))$ $\stackrel{f_i \text{ are } \varphi\text{-contractions}}{\leq}$ $\varphi^{[n]}(d(u, v))$ $\stackrel{\varphi \text{ is increasing}}{\leq}$ $\varphi^{[n]}(d(X))$, so $d(X_{[\alpha]_n}) \leq \varphi^{[n]}(d(X))$, hence $d(\overline{X_{[\alpha]_n}}) \leq \varphi^{[n]}(d(X))$ for every $n \in \mathbb{N}^*$. As (X, d) is complete, making use of Remark

2.2, i) and the fact that $\overline{X_{[\alpha]_{n+1}}} \subseteq \overline{X_{[\alpha]_n}}$ for every $n \in \mathbb{N}^*$, we conclude that the set $\bigcap_{n \in \mathbb{N}} \overline{X_{[\alpha]_n}}$ has one element denoted by a_α , i.e.

$$\bigcap_{n \in \mathbb{N}} \overline{X_{[\alpha]_n}} = \{a_\alpha\}. \quad (1)$$

Let us note that $f_i(a_\alpha) \in f_i(\bigcap_{n \in \mathbb{N}^*} \overline{X_{[\alpha]_n}}) \subseteq \bigcap_{n \in \mathbb{N}^*} f_i(\overline{X_{[\alpha]_n}}) \stackrel{\text{Remark 2.4}}{\subseteq} \bigcap_{n \in \mathbb{N}^*} \overline{f_i(X_{[\alpha]_n})} = \bigcap_{n \in \mathbb{N}^*} \overline{X_{[i\alpha]_n}} \stackrel{(1)}{=} \{a_{i\alpha}\}$, so

$$f_i(a_\alpha) = a_{i\alpha}, \quad (2)$$

for every $i \in I$ and every $\alpha \in \Lambda(I)$. For $\alpha = \alpha_1 \alpha_2 \dots \alpha_n \dots \in \Lambda(I)$ and $n \in \mathbb{N}^*$, with the notation $\beta_n = \alpha_{n+1} \alpha_{n+2} \dots \alpha_m \dots \in \Lambda(I)$, we have $a_\alpha = a_{[\alpha]_n \beta_n} \stackrel{(2)}{=} f_{[\alpha]_n}(a_{\beta_n}) \in X_{[\alpha]_n}$. Hence $\{a_\alpha\} \subseteq \bigcap_{n \in \mathbb{N}^*} X_{[\alpha]_n} \subseteq \bigcap_{n \in \mathbb{N}^*} \overline{X_{[\alpha]_n}} = \{a_\alpha\}$, so $\{a_\alpha\} = \bigcap_{n \in \mathbb{N}^*} \overline{X_{[\alpha]_n}}$.

b) Let us consider $\alpha, \beta \in \Lambda(I)$ such that $a_\alpha \neq a_\beta$. Then Remark 2.2, i) assures the existence of a $n_0 \in \mathbb{N}^*$ such that $\varphi^{[n_0]}(d(X)) < \frac{d(a_\alpha, a_\beta)}{3}$. Consequently, since (as we have seen above) $d(X_{[\alpha]_{n_0}}) \leq \varphi^{[n_0]}(d(X))$ and $d(X_{[\beta]_{n_0}}) \leq \varphi^{[n_0]}(d(X))$, we get $d(X_{[\alpha]_{n_0}}) < \frac{d(a_\alpha, a_\beta)}{3}$ and $d(X_{[\beta]_{n_0}}) < \frac{d(a_\alpha, a_\beta)}{3}$. If, by reductio ad absurdum, $X_{[\alpha]_{n_0}} \cap X_{[\beta]_{n_0}} \neq \emptyset$, then choosing $x \in X_{[\alpha]_{n_0}} \cap X_{[\beta]_{n_0}}$, we get the following contradiction: $d(a_\alpha, a_\beta) \leq d(a_\alpha, x) + d(x, a_\beta) \leq d(X_{[\alpha]_{n_0}}) + d(X_{[\beta]_{n_0}}) < \frac{2d(a_\alpha, a_\beta)}{3}$. \square

Remark 3.2.

i) With the notation $A = \{a_\alpha \mid \alpha \in \Lambda(I)\}$, the function $\pi : \Lambda(I) \rightarrow A$, given by $\pi(\alpha) = a_\alpha$ for every $\alpha \in \Lambda(I)$, is continuous.

Indeed, given a fixed $\alpha \in \Lambda(I)$, as $\lim_{n \rightarrow \infty} d(X_{[\alpha]_n}) = 0$, for every $\varepsilon > 0$ there exists $m \in \mathbb{N}^*$ such that $X_{[\alpha]_m} \subseteq B(a_\alpha, \varepsilon)$, so $B(\alpha, \frac{1}{3^m}) \subseteq \{\omega \in \Lambda(I) \mid [\omega]_m = [\alpha]_m\} \subseteq \pi^{-1}(X_{[\alpha]_m}) \subseteq \pi^{-1}(B(a_\alpha, \varepsilon))$, i.e. $\pi(B(\alpha, \frac{1}{3^m})) \subseteq B(\pi(\alpha), \varepsilon)$.

ii) Considering the function $F_{\mathcal{S}} : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ given by $F_{\mathcal{S}}(C) = \bigcup_{i \in I} f_i(C)$ for every $C \in \mathcal{K}(X)$, using (2) from the proof of Proposition 3.1, we infer that $F_{\mathcal{S}}(A) = A$, i.e., taking into account the uniqueness of the fixed point of $F_{\mathcal{S}}$ (see Theorem 2.5 from [7]), A is the attractor of the iterated function system \mathcal{S} . Moreover, the same result guarantees that $\lim_{n \rightarrow \infty} h(F_{\mathcal{S}}^{[n]}(B), A) = 0$ for every $B \in \mathcal{K}(X)$, where h designates the Hausdorff-Pompeiu metric.

The notion of family of functions having attractor

As $X_{[\alpha]_0} = X$, the above considerations suggest the following:

Definition 3.3. We say that a family of functions $\mathcal{F} = (f_i)_{i \in I}$, where $f_i : X \rightarrow X$ and I is finite, has attractor if the following two properties are valid:

a) For every $\alpha \in \Lambda(I)$, the set $\bigcap_{n \in \mathbb{N}} X_{[\alpha]_n}$ has a unique element which is denoted by a_α .

b) If $a_\alpha \neq a_\beta$, where $\alpha, \beta \in \Lambda(I)$, then there exists $n_0 \in \mathbb{N}$ such that $X_{[\alpha]_{n_0}} \cap X_{[\beta]_{n_0}} = \emptyset$.

The set $A \stackrel{\text{def}}{=} \{a_\alpha \mid \alpha \in \Lambda(I)\}$ is called the attractor of \mathcal{F} .

A metric on the attractor which makes φ -contractions all the functions of a family having attractor

Theorem 3.4. If $\mathcal{F} = (f_i)_{i \in I}$ is a family of functions having attractor A , then there exist a metric d on A and a comparison function φ such that $d(f_i(x), f_i(y)) \leq \varphi(d(x, y))$ for every $i \in I$ and every $x, y \in A$.

Proof. Considering the function $\pi : \Lambda(I) \rightarrow A$, given by $\pi(\alpha) = a_\alpha$ for every $\alpha \in \Lambda(I)$, the binary relation on $\Lambda(I)$, given by $\alpha \sim \beta$ if and only if $\pi(\alpha) = \pi(\beta)$, turns out to be an equivalence relation. We transport the quotient topology on $\Lambda(I)/\sim$ on the topology τ_A on A via the bijection $g : \Lambda(I)/\sim \rightarrow A$ given by $g([\alpha]) = \pi(\alpha)$ for every $[\alpha] \in \Lambda(I)/\sim$.

Note that:

- i) g is a homeomorphism;
- ii) the function $p : \Lambda(I) \rightarrow \Lambda(I)/\sim$, given by $p(\alpha) = [\alpha]$ for every $\alpha \in \Lambda(I)$, is continuous;
- iii) $\pi = g \circ p$ is continuous.

Claim 1. $f_i \circ \pi = \pi \circ \tau_i$ for every $i \in I$.

Justification of claim 1. We have $(f_i \circ \pi)(\alpha) = f_i(a_\alpha) \in f_i(\bigcap_{n \in \mathbb{N}} X_{[\alpha]_n}) \subseteq \bigcap_{n \in \mathbb{N}} f_i(X_{[\alpha]_n}) = \bigcap_{n \in \mathbb{N}} X_{[i\alpha]_n} = \{a_{i\alpha}\} = \{(\pi \circ \tau_i)(\alpha)\}$ for every $i \in I$ and every $\alpha \in \Lambda(I)$.

Note that Claim 1 implies that $A = \bigcup_{i \in I} f_i(A)$.

Claim 2. $f_i : (A, \tau_A) \rightarrow (A, \tau_A)$ is continuous for every $i \in I$.

Justification of claim 2. Taking into account i), it suffices to prove that $f_i \circ g : \Lambda(I) / \sim \rightarrow A$, given by

$$(f_i \circ g)([\alpha]) \stackrel{\text{Claim 1}}{=} (\pi \circ \tau_i)(\alpha), \quad (1)$$

is continuous. Since $\alpha \sim \beta \Leftrightarrow \pi(\alpha) = \pi(\beta) \Rightarrow (f_i \circ \pi)(\alpha) = (f_i \circ \pi)(\beta) \stackrel{\text{Claim 1}}{\Leftrightarrow} (\pi \circ \tau_i)(\alpha) = (\pi \circ \tau_i)(\beta) \stackrel{(1)}{\Leftrightarrow} (f_i \circ g)([\alpha]) = (f_i \circ g)([\beta])$ and the function $h = \pi \circ \tau_i : \Lambda(I) \rightarrow A$, described by $h(\alpha) = (f_i \circ g)([\alpha])$ for every $\alpha \in \Lambda(I)$, is continuous (as a composition of continuous functions; see Remark 2.6 and iii)), relying on Theorem 4.3, page 126, from [6], we get the conclusion.

Claim 3. (A, τ_A) is compact.

Justification of claim 3. From ii) and Remark 2.7, ii), we conclude that $\Lambda(I) / \sim$ is compact. Using i), we get the conclusion.

Claim 4. The set $R = \{(\alpha, \beta) \in \Lambda(I) \times \Lambda(I) \mid \alpha \sim \beta\}$ is closed.

Justification of claim 4. Let us consider $(\alpha, \beta) \in \overline{R}$. Then there exists $((\alpha_n, \beta_n))_{n \in \mathbb{N}} \subseteq R$ such that $\lim_{n \rightarrow \infty} (\alpha_n, \beta_n) = (\alpha, \beta)$ and consequently $\lim_{n \rightarrow \infty} \alpha_n = \alpha$ and $\lim_{n \rightarrow \infty} \beta_n = \beta$. If $a_\alpha \neq a_\beta$, then, according to the property b) from the definition of a family of functions having attractor, there exists $n_0 \in \mathbb{N}$ such that $X_{[\alpha]_{n_0}} \cap X_{[\beta]_{n_0}} = \emptyset$. As $\lim_{n \rightarrow \infty} \alpha_n = \alpha$ and $\lim_{n \rightarrow \infty} \beta_n = \beta$, there exists $n_1 \in \mathbb{N}$ such that $[\alpha_n]_{n_0} = [\alpha]_{n_0}$ and $[\beta_n]_{n_0} = [\beta]_{n_0}$ for every $n \in \mathbb{N}$, $n \geq n_1$ (see Remark 2.7, i)). But $\alpha_n \sim \beta_n$ (because $(\alpha_n, \beta_n) \in R$), i.e. $a_{\alpha_n} = a_{\beta_n}$, and therefore we get the following contradiction: $a_{\alpha_n} = a_{\beta_n} \in X_{[\alpha_n]_{n_0}} \cap X_{[\beta_n]_{n_0}} = X_{[\alpha]_{n_0}} \cap X_{[\beta]_{n_0}} = \emptyset$. Hence $a_\alpha = a_\beta$, i.e. $\alpha \sim \beta$, so $(\alpha, \beta) \in R$. Therefore R is closed.

Claim 5. (A, τ_A) is Hausdorff.

Justification of claim 5. From the compactness of $\Lambda(I)$ (see Remark 2.7, ii)) and Claim 4, we infer that $\Lambda(I) / \sim$ is Hausdorff. Using i) we get the conclusion.

Claims 1, 2, 3 and 5 assure us that $(A, (f_i)_{i \in I})$ is a topological self-similar system and, based on Theorem 2.9, there exist a metric d on A compatible with τ_A and a comparison function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that $d(f_i(x), f_i(y)) \leq \varphi(d(x), d(y))$ for every $i \in I$ and every $x, y \in A$. \square

Let us consider the function $n : X \rightarrow \mathbb{N} \cup \{\infty\}$ given by $n(x) = \sup\{m \in \mathbb{N} \mid x \in F^{[m]}(X)\}$ for every $x \in X$, where $F : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is described by $F(C) = \bigcup_{i \in I} f_i(C)$ for every $C \in \mathcal{P}(X) \stackrel{\text{def}}{=} \{Y \mid Y \subseteq X\}$.

The following result provides an alternative characterization of the attractor A via the function n .

Proposition 3.5. *In the framework of the above theorem, we have $A = \{x \in X \mid n(x) = \infty\}$.*

Proof.

" \subseteq " If $x \in A$, then there exists $\alpha \in \Lambda(I)$ such that $x = a_\alpha$, hence $x \in X_{[\alpha]_m} \subseteq F^{[m]}(X)$ for every $m \in \mathbb{N}$. So $n(x) = \sup\{m \in \mathbb{N} \mid x \in F^{[m]}(X)\} = \sup \mathbb{N} = \infty$.

" \supseteq " Since $n(x) = \sup\{m \in \mathbb{N} \mid x \in F^{[m]}(X)\} = \infty$, for every $m \in \mathbb{N}$ there exists $\alpha_m \in \Lambda_m(I)$ such that $x \in X_{\alpha_m}$. There exists $i_1 \in I$ such that $\{\omega \in \Lambda^*(I) \mid x \in X_{i_1\omega}\}$ is infinite. Indeed, if this is not the case, then the set $M_i \stackrel{def}{=} \{\omega \in \Lambda^*(I) \mid x \in X_{i\omega}\}$ is finite for every $i \in I$. If $m_i \stackrel{def}{=} \max\{|i\omega| \mid \omega \in M_i\}$, then we get the contradiction that there exists no $\alpha \in \Lambda_{m+1}(I)$ such that $x \in X_\alpha$, where $m = 1 + \max\{m_i \mid i \in I\}$. Repeating this procedure we get $\alpha = \alpha_1\alpha_2\dots\alpha_n\dots \in \Lambda(I)$ such that $x \in \bigcap_{n \in \mathbb{N}} X_{[\alpha]_n}$, i.e. $x = a_\alpha \in A$. \square

The family of sets $\{\tilde{X}_\alpha \mid \alpha \in \Lambda^*(I)\}$ associated to a family of functions having attractor

Given a family of functions $\mathcal{F} = (f_i)_{i \in I}$ having attractor A , in the sequel, for $\alpha \in \Lambda^*(I)$ we shall use the following notations:

$$Y_\alpha \stackrel{not}{=} \{a_\beta \mid X_\alpha \cap X_{[\beta]_n} \neq \emptyset \text{ for every } n \in \mathbb{N}\} \text{ and } \tilde{X}_\alpha \stackrel{not}{=} X_\alpha \cup Y_\alpha.$$

Proposition 3.6 (The properties of the sets X_α and Y_α). *In the above framework, we have:*

- a) $A_\alpha \subseteq Y_\alpha \subseteq A$ for every $\alpha \in \Lambda^*(I)$;
- b) $X_\alpha \subseteq \tilde{X}_\alpha \subseteq X_\alpha \cup A$ for every $\alpha \in \Lambda^*(I)$;
- c) $Y_{[\alpha]_{n+1}} \subseteq Y_{[\alpha]_n}$ for every $\alpha \in \Lambda(I)$ and every $n \in \mathbb{N}$;
- d) $\bigcap_{n \in \mathbb{N}} (X_{[\alpha]_n} \cup Y_{[\alpha]_n}) = \left(\bigcap_{n \in \mathbb{N}} X_{[\alpha]_n} \right) \cup \left(\bigcap_{n \in \mathbb{N}} Y_{[\alpha]_n} \right)$ for every $\alpha \in \Lambda(I)$;
- e) $\bigcap_{n \in \mathbb{N}} Y_{[\alpha]_n} = \{a_\alpha\}$ for every $\alpha \in \Lambda(I)$;
- f) $A \cap X_\alpha \subseteq Y_\alpha$ for every $\alpha \in \Lambda^*(I)$;
- g) $f_i(Y_\alpha) \subseteq Y_{i\alpha}$ for every $\alpha \in \Lambda^*(I)$ and every $i \in I$.

Proof.

a) If $z \in A_\alpha$, then there exists $\gamma \in \Lambda(I)$ such that $z = f_\alpha(a_\gamma)$, so $z \in X_\alpha$. Moreover, $z = f_\alpha(a_\gamma) \in f_\alpha(\bigcap_{n \in \mathbb{N}} X_{[\gamma]_n}) \subseteq \bigcap_{n \in \mathbb{N}} f_\alpha(X_{[\gamma]_n}) \subseteq \bigcap_{n \in \mathbb{N}} X_{[\alpha\gamma]_n} = \{a_{\alpha\gamma}\}$, hence $z = a_{\alpha\gamma} \in X_\alpha \cap X_{[\alpha\gamma]_n}$ for every $n \in \mathbb{N}$, i.e. $z \in Y_\alpha$.

b) It results immediately from a).

c) If $z \in Y_{[\alpha]_{n+1}}$, then there exists $\beta \in \Lambda(I)$ such that $z = a_\beta$ and $X_{[\alpha]_{n+1}} \cap X_{[\beta]_k} \neq \emptyset$ for every $k \in \mathbb{N}$. As $X_{[\alpha]_{n+1}} \cap X_{[\beta]_k} \subseteq X_{[\alpha]_n} \cap X_{[\beta]_k}$, we deduce that $X_{[\alpha]_n} \cap X_{[\beta]_k} \neq \emptyset$ for every $k \in \mathbb{N}$, i.e. $z = a_\beta \in Y_{[\alpha]_n}$.

d)

" \supseteq " It is clear.

" \subseteq " Let us suppose that there exists $x \in X_{[\alpha]_n} \cup Y_{[\alpha]_n}$ for every $n \in \mathbb{N}$ such that $x \notin (\bigcap_{n \in \mathbb{N}} X_{[\alpha]_n}) \cup (\bigcap_{n \in \mathbb{N}} Y_{[\alpha]_n})$, i.e. there exist $n_1, n_2 \in \mathbb{N}$ such that $x \notin X_{[\alpha]_{n_1}}$ and $x \notin Y_{[\alpha]_{n_2}}$. Then, in view of c), we have $x \notin X_{[\alpha]_m}$ and $x \notin Y_{[\alpha]_m}$ which leads to the contradiction $x \notin X_{[\alpha]_m} \cup Y_{[\alpha]_m}$, where $m = \max\{n_1, n_2\}$.

e)

" \supseteq " We have $a_\alpha \stackrel{\text{Claim 1 from the proof of Theorem 3.4}}{\in} \bigcap_{n \in \mathbb{N}} A_{[\alpha]_n} \stackrel{\text{a)}}{\subseteq} \bigcap_{n \in \mathbb{N}} Y_{[\alpha]_n}$.

" \subseteq " If $c \in \bigcap_{n \in \mathbb{N}} Y_{[\alpha]_n}$, then there exists $(\beta_n)_{n \in \mathbb{N}} \subseteq \pi^{-1}(\{c\}) \subseteq \Lambda(I)$ such that

$$X_{[\alpha]_n} \cap X_{[\beta_n]_k} \neq \emptyset, \quad (1)$$

for every $n, k \in \mathbb{N}$. The compactness of $\Lambda(I)$ (see Remark 2.7, ii)) assures the existence of a subsequence $(\beta_{n_l})_{l \in \mathbb{N}}$ of $(\beta_n)_{n \in \mathbb{N}}$ and of an element $\beta \in \Lambda(I)$ such that $\lim_{l \rightarrow \infty} \beta_{n_l} = \beta$. As $\pi(\beta_{n_l}) = c$, i.e. $a_{\beta_{n_l}} = c$, and π is continuous (see Remark 3.2, i)), we infer that $\pi(\beta) = c$, i.e. $a_\beta = c$. By replacing β_j with β_{n_l} for all $j \in \{n_{l-1} + 1, \dots, n_l - 1\}$, we can suppose that $\lim_{n \rightarrow \infty} \beta_n = \beta$. Hence for every $l \in \mathbb{N}$ there exists $n_l \in \mathbb{N}$, $n_l > l$ such that

$$[\beta_n]_l = [\beta]_l, \quad (2)$$

for all $n \in \mathbb{N}$, $n \geq n_l$. Hence $X_{[\alpha]_{n_l}} \cap X_{[\beta_{n_l}]_l} \stackrel{(1)}{\neq} \emptyset$, i.e., in view of (2), $X_{[\alpha]_{n_l}} \cap X_{[\beta]_l} \neq \emptyset$ and since $X_{[\alpha]_{n_l}} \subseteq X_{[\alpha]_l}$, we infer that $X_{[\alpha]_l} \cap X_{[\beta]_l} \neq \emptyset$ for every $l \in \mathbb{N}$. Therefore, taking into account the property b) of a family of functions having attractor, we conclude that $a_\alpha = a_\beta = c$.

f) If $z \in A \cap X_\alpha$, then there exists $\gamma \in \Lambda(I)$ such that $z = a_\gamma \in X_{[\gamma]_n}$, so $z \in X_\alpha \cap X_{[\gamma]_n}$ and therefore $X_\alpha \cap X_{[\gamma]_n} \neq \emptyset$ for every $n \in \mathbb{N}$. Consequently $z = a_\gamma \in Y_\alpha$.

g) If $z \in f_i(Y_\alpha)$, then there exists $\beta \in \Lambda(I)$ such that $z = f_i(a_\beta)$ and $X_\alpha \cap X_{[\beta]_n} \neq \emptyset$ for every $n \in \mathbb{N}$. Since $\emptyset \neq f_i(X_\alpha \cap X_{[\beta]_n}) \subseteq f_i(X_\alpha) \cap f_i(X_{[\beta]_n}) = X_{i\alpha} \cap X_{[i\beta]_{n+1}}$ for every $n \in \mathbb{N}$, we conclude that $a_{i\beta}$ Claim 1 from the proof of Theorem 3.4
 $f_i(a_\beta) = z \in Y_{i\alpha}$. \square

Proposition 3.7 (The properties of the sets \tilde{X}_α). *In the above framework, we have:*

- a) $X_{[\alpha]_{n+1}} \subseteq \tilde{X}_{[\alpha]_n}$ for every $\alpha \in \Lambda(I)$ and every $n \in \mathbb{N}$;
- b) $\bigcap_{n \in \mathbb{N}} \tilde{X}_\alpha = \{a_\alpha\}$ for every $\alpha \in \Lambda(I)$.
- c) $a_\beta \in \tilde{X}_\alpha$, provided that $X_\alpha \cap X_{[\beta]_n} \neq \emptyset$ for every $n \in \mathbb{N}$, where $\alpha \in \Lambda^*(I)$ and $\beta \in \Lambda(I)$.
- d) for every $a_\alpha, a_\beta \in A$ such that $a_\alpha \neq a_\beta$, there exists $n_0 \in \mathbb{N}$ having the property that $X_{[\alpha]_{n_0}} \cap X_{[\beta]_{n_0}} = \emptyset$.
- e) $f_i(\tilde{X}_\alpha) \subseteq \tilde{X}_{i\alpha}$ for every $i \in I$ and every $\alpha \in \Lambda^*(I)$.

Proof.

a) As $X_{[\alpha]_{n+1}} \subseteq X_{[\alpha]_n}$ and $Y_{[\alpha]_{n+1}} \subseteq Y_{[\alpha]_n}$ Proposition 3.6, c), we infer that $X_{[\alpha]_{n+1}} \cup Y_{[\alpha]_{n+1}} \subseteq X_{[\alpha]_n} \cup Y_{[\alpha]_n}$, i.e. $X_{[\alpha]_{n+1}} \subseteq \tilde{X}_{[\alpha]_n}$.

b) We have $\bigcap_{n \in \mathbb{N}} \tilde{X}_{[\alpha]_n} = \bigcap_{n \in \mathbb{N}} (X_{[\alpha]_n} \cup Y_{[\alpha]_n})$ Proposition 3.6, d) $= (\bigcap_{n \in \mathbb{N}} X_{[\alpha]_n}) \cup (\bigcap_{n \in \mathbb{N}} Y_{[\alpha]_n})$ Proposition 3.6, e) $= \{a_\alpha\}$.

c) Since $\tilde{X}_\alpha \cap \tilde{X}_{[\beta]_l} \neq \emptyset$, i.e. $(X_\alpha \cup Y_\alpha) \cap (X_{[\beta]_l} \cup Y_{[\beta]_l}) \neq \emptyset$, we get $(X_\alpha \cap X_{[\beta]_l}) \cup (X_\alpha \cap Y_{[\beta]_l}) \cup (Y_\alpha \cap X_{[\beta]_l}) \cup (Y_\alpha \cap Y_{[\beta]_l}) \neq \emptyset$ for every $l \in \mathbb{N}$. Thus, at least one of the sets $\{l \in \mathbb{N} \mid X_\alpha \cap X_{[\beta]_l} \neq \emptyset\}$, $\{l \in \mathbb{N} \mid X_\alpha \cap Y_{[\beta]_l} \neq \emptyset\}$, $\{l \in \mathbb{N} \mid Y_\alpha \cap X_{[\beta]_l} \neq \emptyset\}$ and $\{l \in \mathbb{N} \mid Y_\alpha \cap Y_{[\beta]_l} \neq \emptyset\}$ is infinite.

If $\{l \in \mathbb{N} \mid X_\alpha \cap X_{[\beta]_l} \neq \emptyset\}$ is infinite, then, as $X_{[\beta]_{l+1}} \subseteq X_{[\beta]_l}$ for every $l \in \mathbb{N}$, we infer that $X_\alpha \cap X_{[\beta]_l} \neq \emptyset$ for every $l \in \mathbb{N}$, so $a_\beta \in Y_\alpha \subseteq X_\alpha \cup Y_\alpha = \tilde{X}_\alpha$.

Since $X_\alpha \cap Y_{[\beta]_l}$ Proposition 3.6, a) $= X_\alpha \cap Y_{[\beta]_l} \cap A = (X_\alpha \cap A) \cap Y_{[\beta]_l}$ Proposition 3.6, f) $\subseteq Y_\alpha \cap Y_{[\beta]_l}$ and $Y_\alpha \cap X_{[\beta]_l}$ Proposition 3.6, a) $= Y_\alpha \cap X_{[\beta]_l} \cap A = Y_\alpha \cap (X_{[\beta]_l} \cap A)$ Proposition 3.6, f) $\subseteq Y_\alpha \cap Y_{[\beta]_l}$ for every $l \in \mathbb{N}$, we deduce that if one of the sets $\{l \in \mathbb{N} \mid X_\alpha \cap Y_{[\beta]_l} \neq \emptyset\}$, $\{l \in \mathbb{N} \mid Y_\alpha \cap X_{[\beta]_l} \neq \emptyset\}$ and $\{l \in \mathbb{N} \mid Y_\alpha \cap Y_{[\beta]_l} \neq \emptyset\}$

is infinite, then, in view of Proposition 3.6, c), we have

$$Y_\alpha \cap Y_{[\beta]_l} \neq \emptyset, \quad (1)$$

for every $l \in \mathbb{N}$.

We are going to prove that $a_\beta \in Y_\alpha \subseteq X_\alpha \cup Y_\alpha = \tilde{X}_\alpha$ and this will closed the justification of c).

From (1) we deduce that, for every $n \in \mathbb{N}$, there exists $a_n \in Y_\alpha \cap Y_{[\beta]_n} \neq \emptyset$. Consequently we can find $\gamma_n, \gamma'_n \in \Lambda(I)$ such that

$$a_n = a_{\gamma_n} = a_{\gamma'_n} \quad (2)$$

and

$$X_\alpha \cap X_{[\gamma_n]_l} \neq \emptyset \text{ and } X_{[\beta]_n} \cap X_{[\gamma'_n]_l} \neq \emptyset, \quad (3)$$

for every $l \in \mathbb{N}$. The compactness of $\Lambda(I)$ (see Remark 2.7, i)) assures the existence of the subsequences $(\gamma_{n_k})_{k \in \mathbb{N}}$ of $(\gamma_n)_{n \in \mathbb{N}}$ and $(\gamma'_{n_k})_{k \in \mathbb{N}}$ of $(\gamma'_n)_{n \in \mathbb{N}}$ and of the elements $\gamma_0, \gamma'_0 \in \Lambda(I)$ such that $\lim_{k \rightarrow \infty} \gamma_{n_k} = \gamma_0$ and $\lim_{k \rightarrow \infty} \gamma'_{n_k} = \gamma'_0$.

By replacing γ_n with γ_{n_k} for all $n \in \{n_{k-1} + 1, \dots, n_k - 1\}$ and γ'_n with γ'_{n_k} for all $n \in \{n_{k-1} + 1, \dots, n_k - 1\}$, we can suppose that $\lim_{n \rightarrow \infty} \gamma_n = \gamma_0$ and $\lim_{n \rightarrow \infty} \gamma'_n = \gamma'_0$. Hence for every $l \in \mathbb{N}$ there exists $n_l \in \mathbb{N}$, $n_l > l$ such that

$$[\gamma_n]_l = [\gamma_0]_l \text{ and } [\gamma'_n]_l = [\gamma'_0]_l, \quad (4)$$

for every $n \in \mathbb{N}$, $n \geq n_l$. Therefore, since $\emptyset \neq X_\alpha \cap X_{[\gamma_n]_l} \stackrel{(3)}{=} X_\alpha \cap X_{[\gamma_0]_l} \stackrel{(4)}{=} X_\alpha \cap X_{[\gamma_0]_l}$ for every $l \in \mathbb{N}$, we get that

$$a_{\gamma_0} \in Y_\alpha. \quad (5)$$

Moreover, since $\emptyset \neq X_{[\beta]_{n_l}} \cap X_{[\gamma'_{n_l}]_l} \subseteq X_{[\beta]_l} \cap X_{[\gamma'_{n_l}]_l} \stackrel{(4)}{=} X_{[\beta]_l} \cap X_{[\gamma'_0]_l}$ for every $l \in \mathbb{N}$, taking into account the property b) of a family of functions having attractor, we conclude that

$$a_{\gamma'_0} = a_\beta. \quad (6)$$

Making use of the continuity of π we get that $\lim_{n \rightarrow \infty} a_{\gamma_n} = a_{\gamma_0}$ and $\lim_{n \rightarrow \infty} a_{\gamma'_n} = a_{\gamma'_0}$ and, taking into account (2), we conclude that $a_\beta \stackrel{(6)}{=} a_{\gamma'_0} = a_{\gamma_0} \stackrel{(5)}{\in} Y_\alpha$.

d) If by reductio ad absurdum, we suppose that $X_{[\alpha]_n} \cap X_{[\beta]_n} \neq \emptyset$ for every $n \in \mathbb{N}$, then we have $\emptyset \neq X_{[\alpha]_{\max\{k,l\}}} \cap X_{[\beta]_{\max\{k,l\}}} \stackrel{a)}{\subseteq} X_{[\alpha]_k} \cap X_{[\beta]_l}$, hence

$\tilde{X}_{[\alpha]_k} \cap \tilde{X}_{[\beta]_l} \neq \emptyset$, for every $k, l \in \mathbb{N}$, so, based on c), we get that $a_\beta \in \tilde{X}_{[\alpha]_k}$ for every $k \in \mathbb{N}$. Using b) we arrive to the contradiction that $a_\alpha = a_\beta$.

e) We have $f_i(\tilde{X}_\alpha) = f_i(X_\alpha \cup Y_\alpha) = f_i(X_\alpha) \cup f_i(Y_\alpha) = X_{i\alpha} \cup f_i(Y_\alpha)$
Proposition 3.6, g)
 $\subseteq X_{i\alpha} \cup Y_{i\alpha} = \tilde{X}_{i\alpha}$. \square

The semi-metric d^μ associated to a decreasing sequence μ and to a family of functions having attractor

Given a family of functions $\mathcal{F} = (f_i)_{i \in I}$ having attractor and a sequence $\mu = (z_n)_{n \in \mathbb{N}}$ such that $0 < z_{n+1} \leq z_n$ for every $n \in \mathbb{N}$, we consider the function $d^\mu : X \times X \rightarrow [0, \infty)$ described by $d^\mu(x, y) = \begin{cases} 0, & x = y, \\ \inf M_{x,y}, & x \neq y \end{cases}$,

where $M_{x,y} = \left\{ \sum_{i=0}^n z_{|\alpha_i|} \mid \text{there exist } n \in \mathbb{N} \text{ and } \alpha_0, \alpha_1, \dots, \alpha_n \in \Lambda^*(I) \text{ such that } x \in \tilde{X}_{\alpha_0}, y \in \tilde{X}_{\alpha_n} \text{ and } \tilde{X}_{\alpha_i} \cap \tilde{X}_{\alpha_{i+1}} \neq \emptyset \text{ for every } i \in \{0, 1, \dots, n-1\} \right\}$.

Proposition 3.8 (The properties of d^μ). *In the above framework, we have:*

- a) $d^\mu(x, x) = 0$ for every $x \in X$;
- b) $d^\mu(x, y) = d^\mu(y, x)$ for every $x, y \in X$;
- c) $d^\mu(x, y) \leq d^\mu(x, z) + d^\mu(z, y)$ for every $x, y, z \in X$;
- d) $d^\mu(x, y) > 0$ for every $x \in X \setminus A$ and every $y \in X \setminus \{x\}$;
- e) $d^\mu(f_i(x), f_i(y)) \leq d^\mu(x, y)$ for every $x, y \in X$;
- f) $d^\mu(x, y) \leq z_0$ for every $x, y \in X$;
- g) If the sequence μ is constant, then d^μ is a metric.

Proof.

a) and b) are obvious, while c) is clear since every chain from x to z and every chain from z to y generate a chain from x to y .

d) Considering the function $m : X \rightarrow \mathbb{N} \cup \{\infty\}$, given by $m(u) = \sup\{|\alpha| \mid \alpha \in \Lambda^*(I) \text{ and } u \in \tilde{X}_\alpha\}$ for every $u \in X$, using a similar argument as in the one used in the proof of Proposition 3.5, we obtain that $m(u) = \infty$ if and only if $u \in A$. Hence $m(x) \in \mathbb{N}$ since $x \in X \setminus A$ and if for $n \in \mathbb{N}$ and $\alpha_0, \alpha_1, \dots, \alpha_n \in \Lambda^*(I)$ we have $x \in \tilde{X}_{\alpha_0}$, $y \in \tilde{X}_{\alpha_n}$ and $\tilde{X}_{\alpha_i} \cap \tilde{X}_{\alpha_{i+1}} \neq \emptyset$ for every $i \in \{0, \dots, n-1\}$, then $z_{m(x)} \leq z_{|\alpha_0|} \leq \sum_{i=0}^n z_{|\alpha_i|}$. So $z_{m(x)}$ is a lower bound for $M_{x,y}$ and, consequently, $0 < z_{m(x)} \leq \inf M_{x,y} = d^\mu(x, y)$.

e) The inequality is obvious if $f_i(x) = f_i(y)$ (in particular, if $x = y$). Otherwise, if for $n \in \mathbb{N}$ and $\alpha_0, \alpha_1, \dots, \alpha_n \in \Lambda^*(I)$ we have $x \in \tilde{X}_{\alpha_0}$, $y \in \tilde{X}_{\alpha_n}$ and $\tilde{X}_{\alpha_j} \cap \tilde{X}_{\alpha_{j+1}} \neq \emptyset$ for every $j \in \{0, 1, \dots, n-1\}$, then $f_i(x) \in f_i(\tilde{X}_{\alpha_0}) \stackrel{\text{Proposition 3.7, e)}}{\subseteq} \tilde{X}_{i\alpha_0}$, $f_i(y) \in f_i(\tilde{X}_{\alpha_n}) \stackrel{\text{Proposition 3.7, e)}}{\subseteq} \tilde{X}_{i\alpha_n}$ and $\emptyset \neq f_i(\tilde{X}_{\alpha_j} \cap \tilde{X}_{\alpha_{j+1}}) \subseteq f_i(\tilde{X}_{\alpha_j}) \cap f_i(\tilde{X}_{\alpha_{j+1}}) \stackrel{\text{Proposition 3.7, e)}}{\subseteq} \tilde{X}_{i\alpha_j} \cap \tilde{X}_{i\alpha_{j+1}}$, so $\tilde{X}_{i\alpha_j} \cap \tilde{X}_{i\alpha_{j+1}} \neq \emptyset$ for every $j \in \{0, 1, \dots, n-1\}$, i.e. $\sum_{j=0}^n z_{|i\alpha_j|} \in M_{f_i(x), f_i(y)}$.

Hence $d^\mu(f_i(x), f_i(y)) = \inf M_{f_i(x), f_i(y)} \leq \sum_{j=0}^n z_{|i\alpha_j|} \leq \sum_{j=0}^n z_{|\alpha_j|}$, i.e. $d^\mu(f_i(x), f_i(y))$ is a lower bound for $M_{x,y}$, so $d^\mu(f_i(x), f_i(y)) \leq \inf M_{x,y} = d^\mu(x, y)$.

f) and g) result from the definition of d^μ . \square

Remark 3.9.

- i) d^μ is a semi-metric.
- ii) From the proof of d) we get that $\{y \in X \mid d^\mu(x, y) < \frac{z_{m(x)}}{2}\} = \{x\}$ for every $x \in X \setminus A$. In other words, the topology generated by d^μ on $X \setminus A$ is the discrete one.
- iii) From e) we conclude that (with respect to d^μ) each of the functions f_i has the Lipschitz constant less or equal to 1.

Given a natural number N , a family of functions $\mathcal{F} = (f_i)_{i \in I}$ having attractor and a sequence $\mu = (z_n)_{n \in \mathbb{N}}$ such that $0 < z_{n+1} \leq z_n$ for every $n \in \mathbb{N}$, we consider the function $d_N^\mu : X \times X \rightarrow [0, \infty)$ described by

$$d_N^\mu(x, y) = \begin{cases} 0, & x = y \\ \inf M_{x,y}^N, & x \neq y \end{cases}, \text{ where } M_{x,y}^N = \left\{ \sum_{i=0}^n z_{|\alpha_i|} \mid \text{there exist } n \in \mathbb{N} \text{ and } \right.$$

$\alpha_0, \alpha_1, \dots, \alpha_n \in \Lambda_0(I) \cup \Lambda_1(I) \cup \dots \cup \Lambda_N(I)$ such that $x \in \tilde{X}_{\alpha_0}$, $y \in \tilde{X}_{\alpha_n}$ and $\tilde{X}_{\alpha_i} \cap \tilde{X}_{\alpha_{i+1}} \neq \emptyset$ for every $i \in \{0, 1, \dots, n-1\}$.

We also consider the sequences μ_N and $\mu_{N,p}$ given by $\mu_N = (z_n^N)_{n \in \mathbb{N}}$ and $\mu_{N,p} = (z_n^{N,p})_{n \in \mathbb{N}}$, where $p \in \mathbb{N}$, $z_n^N = \begin{cases} z_n, & n \in \{0, 1, \dots, N\} \\ z_N, & n \in \mathbb{N}, n \geq N+1 \end{cases}$ and

$$z_n^{N,p} = \begin{cases} z_n, & n \in \{0, 1, \dots, N\} \\ z_N, & n \in \{N+1, \dots, N+p\} \\ \frac{z_N}{2}, & n \in \mathbb{N}, n \geq N+p+1 \end{cases}.$$

Proposition 3.10 (The properties of d_N^μ). *In the above framework, we have:*

- a) $d^\mu \leq d_{N+1}^\mu \leq d_N^\mu$ for every $N \in \mathbb{N}$;
- b) $d^\mu = \lim_{N \rightarrow \infty} d_N^\mu = \inf_{N \in \mathbb{N}} d_N^\mu$ (i.e. $\lim_{N \rightarrow \infty} d_N^\mu(x, y) = d^\mu(x, y)$ for every $x, y \in X$);
- c) $d^{\mu_{N,p}} \leq d^{\mu_{N,p+1}} \leq d^{\mu_N}$ for every $N, p \in \mathbb{N}$;
- d) $d_N^\mu = d^{\mu_N}$ for every $N \in \mathbb{N}$.

Proof.

a) It results from the inclusion $M_{x,y}^N \subseteq M_{x,y}^{N+1} \subseteq M_{x,y}$ which is valid for every $x, y \in X$ and every $N \in \mathbb{N}$.

b) Given $x, y \in X$, $x \neq y$, for every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and $\alpha_0, \alpha_1, \dots, \alpha_n \in \Lambda^*(I)$ such that $x \in \tilde{X}_{\alpha_0}$, $y \in \tilde{X}_{\alpha_n}$ and $\tilde{X}_{\alpha_i} \cap \tilde{X}_{\alpha_{i+1}} \neq \emptyset$ for every $i \in \{0, 1, \dots, n-1\}$ and $\sum_{i=0}^n z_{|\alpha_i|} < d^\mu(x, y) + \varepsilon$. As $\sum_{i=0}^n z_{|\alpha_i|} \in M_{x,y}^{N_\varepsilon}$, where $N_\varepsilon = \max\{|\alpha_0|, |\alpha_1|, \dots, |\alpha_n|\}$, we infer that $d_N^\mu \leq d_{N_\varepsilon}^\mu = \inf M_{x,y}^{N_\varepsilon} \leq \sum_{i=0}^n z_{|\alpha_i|} < d^\mu(x, y) + \varepsilon$, so $|d_N^\mu - d^\mu| < \varepsilon$ for every $N \in \mathbb{N}$, $N \geq N_\varepsilon$. Hence $\lim_{N \rightarrow \infty} d_N^\mu(x, y) = d^\mu(x, y)$ for every $x, y \in X$, $x \neq y$. The equality is obvious for $x = y$.

c) Let us note that if the sequences $\mu = (z_n)_{n \in \mathbb{N}}$ and $\nu = (t_n)_{n \in \mathbb{N}}$ have the property $z_n \leq t_n$ for every $n \in \mathbb{N}$ (we denote this situation by $\mu \prec \nu$), then $d^\mu \leq d^\nu$. Now the conclusion results from the fact that $\mu_{N,p} \prec \mu_{N,p+1} \prec \mu_N$.

d) First let us note that

$$d_N^\mu = d_N^{\mu_N}. \quad (1)$$

Moreover

$$d_N^{\mu_N} = d_M^{\mu_N}, \quad (2)$$

for every $M \in \mathbb{N}$, $M > N$.

Indeed, we have $d_M^{\mu_N} \stackrel{a)}{\leq} d_N^{\mu_N}$ for every $M \in \mathbb{N}$, $M > N$, it remains to prove that $d_N^{\mu_N} \leq d_M^{\mu_N}$ for every $M \in \mathbb{N}$, $M > N$. This follows from the fact that if for $x, y \in X$, $x \neq y$ there exist $n \in \mathbb{N}$ and $\alpha_0, \alpha_1, \dots, \alpha_n \in \Lambda_0(I) \cup \Lambda_1(I) \cup \dots \cup \Lambda_M(I)$ such that $x \in \tilde{X}_{\alpha_0}$, $y \in \tilde{X}_{\alpha_n}$ and $\tilde{X}_{\alpha_i} \cap \tilde{X}_{\alpha_{i+1}} \neq \emptyset$ for every $i \in \{0, 1, \dots, n-1\}$, then $x \in \tilde{X}_{N[\alpha_0]}$, $y \in \tilde{X}_{N[\alpha_n]}$, $\emptyset \neq \tilde{X}_{\alpha_i} \cap \tilde{X}_{\alpha_{i+1}} \stackrel{\text{Proposition 3.7, a)}}{\subseteq} \tilde{X}_{N[\alpha_i]} \cap \tilde{X}_{N[\alpha_{i+1}]}$ (so $\tilde{X}_{N[\alpha_i]} \cap \tilde{X}_{N[\alpha_{i+1}]} \neq \emptyset$) for every $i \in \{0, 1, \dots, n-1\}$.

$\{0, 1, \dots, n-1\}$ and $\sum_{i=0}^n z_{|\alpha_i|} = \sum_{i=0}^n z_{|N[\alpha_i]|}$, where $N[\alpha] = \begin{cases} \alpha, & \text{if } |\alpha| \leq N \\ [\alpha]_N, & \text{if } |\alpha| > N \end{cases}$.

Based on b), by passing to limit as M goes to ∞ in (2), and using (1), we get the conclusion. \square

Proposition 3.11. *In the above framework, we have $\lim_{p \rightarrow \infty} d^{\mu_{N,p}} = d^{\mu_N}$ (i.e. $\lim_{p \rightarrow \infty} d^{\mu_{N,p}}(x, y) = d^{\mu_N}(x, y)$ for every $x, y \in X$) for every $N \in \mathbb{N}$.*

Proof. Note that $\lim_{p \rightarrow \infty} d^{\mu_{N,p}}(x, y)$ exists and is finite since, according to Proposition 3.10, c) the sequence $(d^{\mu_{N,p}}(x, y))_{p \in \mathbb{N}}$ is increasing and bounded for every $x, y \in X$.

If $d^{\mu_N}(x, y) = 0$, then, based on Proposition 3.10, c), we get that $\lim_{p \rightarrow \infty} d^{\mu_{N,p}}(x, y) = d^{\mu_N}(x, y)$.

Hence we have to consider only the case when $d^{\mu_N}(x, y) \neq 0$. Taking into account Proposition 3.10, c), we have $\lim_{p \rightarrow \infty} d^{\mu_{N,p}}(x, y) \leq d^{\mu_N}(x, y)$. Let us suppose, by reductio ad absurdum, that $l_0 \stackrel{\text{not}}{=} \lim_{p \rightarrow \infty} d^{\mu_{N,p}}(x, y) = \sup_{p \in \mathbb{N}} d^{\mu_{N,p}}(x, y) < d^{\mu_N}(x, y)$. Then $d^{\mu_{N,p}}(x, y) \leq l_0 < l \stackrel{\text{not}}{=} \frac{l_0 + d^{\mu_N}(x, y)}{2} < d^{\mu_N}(x, y)$ for every $p \in \mathbb{N}$. Hence there exist $n_p \in \mathbb{N}$ and $\alpha_0^p, \alpha_1^p, \dots, \alpha_{n_p}^p \in \Lambda^*(I)$ such that $x \in \tilde{X}_{\alpha_0^p}$, $y \in \tilde{X}_{\alpha_{n_p}^p}$, $\tilde{X}_{\alpha_i^p} \cap \tilde{X}_{\alpha_{i+1}^p} \neq \emptyset$ for every $i \in \{0, 1, \dots, n_p - 1\}$ and $\sum_{i=0}^{n_p} z_{|\alpha_i^p|}^{N,p} < l$. Since $z_{|\alpha_i^p|}^{N,p} \geq \frac{z_N}{2}$ for every $i \in \{0, 1, \dots, n_p\}$, we infer that $(n_p + 1) \frac{z_N}{2} < l$ for every $p \in \mathbb{N}$, so the sequence $(n_p)_{p \in \mathbb{N}} \subseteq \mathbb{N}$ is bounded and therefore there exists a subsequence $(n_{p_k})_{k \in \mathbb{N}}$ of $(n_p)_{p \in \mathbb{N}}$ such that $n_{p_1} = n_{p_2} = \dots \stackrel{\text{not}}{=} m$.

We say that $i \in \{0, 1, \dots, m\}$ is:

- of type I if $\overline{\lim_{k \rightarrow \infty}} |\alpha_i^{p_k}| < \infty$;

- of type II if $\overline{\lim_{k \rightarrow \infty}} |\alpha_i^{p_k}| = \infty$.

If i is of type I , then there exists $C \in \mathbb{R}$ such that $|\alpha_i^{p_k}| < C$ for every $k \in \mathbb{N}$, so, by passing to a subsequence, we can assume that $\alpha_i^{p_1} = \alpha_i^{p_2} = \dots = \alpha_i^{p_n} = \dots \stackrel{\text{not}}{=} \alpha_i$.

If i is of type II , then, by passing to a subsequence, we can assume that:

i) $\lim_{k \rightarrow \infty} |\alpha_i^{p_k}| = \infty$;

ii) $|\alpha_i^{p_k}| < |\alpha_i^{p_{k+1}}|$ for every $k \in \mathbb{N}$;

iii) there exists $\alpha_i \in \Lambda(I)$ such that $[\alpha_i]_{|\alpha_i^{p_k}|} = \alpha_i^{p_k}$ for every $k \in \mathbb{N}$ (since there exists $j_1 \in I$ which is the first letter for an infinity of $\alpha_i^{p_k}$ -otherwise, we contradict i)- and we choose j_1 to be the first letter of α_i ; the same argument provides $j_2 \in I$ which is the second letter for an infinity of $\alpha_i^{p_k}$ having j_1 as the first letter and we choose j_2 to be the second letter of α_i ; we continue this procedure).

For a fixed $j \in \{0, 1, \dots, m-1\}$ the following four cases are possible:

- a) j and $j+1$ are of type I ;
- b) j is of type I and $j+1$ is of type II ;
- c) j is of type II and $j+1$ is of type I ;
- d) j and $j+1$ are of type II .

In case a) we have

$$\tilde{X}_{\alpha_j} \cap \tilde{X}_{\alpha_{j+1}} \neq \emptyset. \quad (1)$$

In case b) we have $\tilde{X}_{\alpha_j} \cap \tilde{X}_{[\alpha_{j+1}]_{|\alpha_{j+1}^{p_k}|}} \neq \emptyset$ for every $k \in \mathbb{N}$, so, according to Proposition 3.7, c), we get

$$a_{\alpha_{j+1}} \in \tilde{X}_{\alpha_j}. \quad (2)$$

In case c), as above, we get

$$a_{\alpha_j} \in \tilde{X}_{\alpha_{j+1}}. \quad (3)$$

In case d), we have $\tilde{X}_{[\alpha_j]_{|\alpha_j^{p_k}|}} \cap \tilde{X}_{[\alpha_{j+1}]_{|\alpha_{j+1}^{p_k}|}} \neq \emptyset$ for every $k \in \mathbb{N}$, so, using Proposition 3.7, d), we obtain that

$$a_{\alpha_j} = a_{\alpha_{j+1}}. \quad (4)$$

First let us note that if all $i \in \{0, 1, \dots, m\}$ would be of type II , then $x \in \tilde{X}_{\alpha_0^{p_k}} = \tilde{X}_{[\alpha_0]_{|\alpha_0^{p_k}|}}$ for every $k \in \mathbb{N}$, so taking into account Proposition 3.7, b), we get that $x = a_{\alpha_0}$. In the same way we obtain that $y = a_{\alpha_m}$ and, based on (4), we conclude that $x = a_{\alpha_0} = a_{\alpha_1} = \dots = a_{\alpha_{m-1}} = a_{\alpha_m} = y$ which contradicts our assumption that $d^{\mu_N}(x, y) \neq 0$. Hence we can assume that at least one $i \in \{0, 1, \dots, m\}$ is of type I .

Now we mention the following four facts:

Fact 1. As we have seen before, if $i \in \{0, 1, \dots, m\}$ is of type II and $u \in \tilde{X}_{\alpha_i^{p_k}}$ for every $k \in \mathbb{N}$, then $u = a_{\alpha_i}$.

Fact 2. If j and q are of type I and $j+1, j+2, \dots, q-1$ are of type II , where $0 \leq j < q \leq m$, then, $a_{\alpha_{j+1}} \stackrel{(2)}{\in} \tilde{X}_{\alpha_j}$, $a_{\alpha_{q-1}} \stackrel{(3)}{\in} \tilde{X}_{\alpha_q}$ and, based on (4), we have $a_{\alpha_{j+1}} = a_{\alpha_{j+2}} = \dots = a_{\alpha_{q-1}}$, so $\tilde{X}_{\alpha_j} \cap \tilde{X}_{\alpha_q} \neq \emptyset$.

Fact 3. If $j, j+1, \dots, q-1$ are of type II and q is of type I , where $0 \leq j < q \leq m$, then based on (4), we have $a_{\alpha_j} = a_{\alpha_{j+1}} = \dots = a_{\alpha_{q-1}}$ and $a_{\alpha_{q-1}} \stackrel{(3)}{\in} \tilde{X}_{\alpha_q}$, so $a_{\alpha_j} \in \tilde{X}_{\alpha_q}$.

Fact 4. If j is of type I and $j+1, \dots, q$ are of type II , where $0 \leq j < q \leq m$, then based on (4), we have $a_{\alpha_{j+1}} = a_{\alpha_{j+2}} = \dots = a_{\alpha_q}$ and $a_{\alpha_{j+1}} \stackrel{(3)}{\in} \tilde{X}_{\alpha_j}$, so $a_{\alpha_q} \in \tilde{X}_{\alpha_j}$.

In view of the above mentioned four facts, we can pick up from the set $\{\alpha_0, \alpha_1, \dots, \alpha_m\}$ a subset $\{\alpha_{i_0} \stackrel{not}{=} \beta_0, \alpha_{i_1} \stackrel{not}{=} \beta_1, \dots, \alpha_{i_l} \stackrel{not}{=} \beta_l\}$, where i_0, i_1, \dots, i_l are all the type I elements of $\{0, 1, \dots, m\}$, such that $x \in \tilde{X}_{\beta_0}$, $y \in \tilde{X}_{\beta_l}$ and $\tilde{X}_{\beta_j} \cap \tilde{X}_{\beta_{j+1}} \neq \emptyset$ for every $j \in \{0, 1, \dots, l-1\}$. Then we get the following

contradiction: $d^{\mu_N}(x, y) \leq \sum_{j=0}^l z_{|\beta_j|}^N \leq \sum_{i=0}^{n_{p_k}} z_{|\alpha_i^{p_k}|}^N = \sum_{i=0}^{n_{p_k}} z_{|\alpha_i^{p_k}|}^{N, n_{p_k}} < l < d^{\mu_N}(x, y)$,

where k is chosen such that $N + p_k > \max\{|\alpha_0|, |\alpha_1|, \dots, |\alpha_m|\}$. \square

Proposition 3.12. *In the above framework, for every $x, y \in X$, $x \neq y$ and $M > 0$, there exists a decreasing sequence $\mu = (z_n)_{n \in \mathbb{N}}$ such that:*

- i) $\lim_{n \rightarrow \infty} z_n = 0$;
- ii) $d^\mu(x, y) > 0$;
- iii) $d^\mu \leq M$.

Proof. For $M > 0$, let us consider the sequence μ^0 , where $\mu^0 = (y_n)_{n \in \mathbb{N}}$ and $y_n = M$ for every $n \in \mathbb{N}$. Note that $d^{\mu^0}(x, y) = M$. By mathematical induction we construct a sequence $(\mu^k)_{k \in \mathbb{N}}$ of sequences such that

$$d^{\mu^k}(x, y) - d^{\mu^{k+1}}(x, y) < \frac{M}{2^{k+2}}, \quad (1)$$

for every $k \in \mathbb{N}$. In fact we construct a strictly increasing sequence $(p_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ such that $\mu^{k+1} = \mu_{p_k+1, p_{k+1}-p_k}^k$, where if p_k is constructed, p_{k+1} is chosen such that (1) is valid based on the fact that $\lim_{p \rightarrow \infty} d^{\mu_{p_k+1, p}^k} = d^{\mu_{p_k+1}^k}$ (see Proposition 3.11). Note that

$$\mu^{k+1} \prec \mu^k, \quad (2)$$

for every $k \in \mathbb{N}$. Since $\|\mu^{k+1} - \mu^k\| \leq \frac{M}{2^{k+1}}$ for every $k \in \mathbb{N}$ (here, for a sequence $(a_n)_{n \in \mathbb{N}}$, by $\|(a_n)_{n \in \mathbb{N}}\|$ we mean $\sup_{n \in \mathbb{N}} |a_n|$), we infer that the sequence $(\mu^k)_{k \in \mathbb{N}}$ is Cauchy, so it is convergent and therefore there exists a sequence $\mu = (z_n)_{n \in \mathbb{N}}$ such that $\mu = \lim_{k \rightarrow \infty} \mu^k$.

$$\text{Let us note that we have } z_n = \begin{cases} M, & n \in \{0, 1, \dots, p_1\} \\ \frac{M}{2}, & n \in \{p_1 + 1, \dots, p_2\} \\ \frac{M}{2^2}, & n \in \{p_2 + 1, \dots, p_3\} \\ \dots & \dots \\ \frac{M}{2^q}, & n \in \{p_q + 1, \dots, p_{q+1}\} \\ \dots & \dots \end{cases} \quad \text{and, if}$$

$$\mu^k = (z_n^k)_{n \in \mathbb{N}}, \text{ then } z_n^k = \begin{cases} M, & n \in \{0, 1, \dots, p_1\} \\ \frac{M}{2}, & n \in \{p_1 + 1, \dots, p_2\} \\ \dots & \dots \\ \frac{M}{2^{k-1}}, & n \in \{p_{k-1} + 1, \dots, p_k\} \\ \frac{M}{2^k}, & n \geq p_k + 1 \end{cases} \quad \text{for every } k \in \mathbb{N}.$$

Now we prove that the decreasing sequence μ satisfies the conditions i), ii) and iii).

- i) is obvious having in view the description of the general term of μ .
- ii) First of all let us note that

$$d^\mu(x_1, y_1) = \lim_{k \rightarrow \infty} d^{\mu^k}(x_1, y_1) = \inf_{k \in \mathbb{N}} d^{\mu^k}(x_1, y_1), \quad (3)$$

for every $x_1, y_1 \in X$.

Indeed, let us fix $x_1, y_1 \in X$. For every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and $\alpha_0, \alpha_1, \dots, \alpha_n \in \Lambda^*(I)$ such that $x_1 \in \tilde{X}_{\alpha_0}$, $y_1 \in \tilde{X}_{\alpha_n}$, $\tilde{X}_{\alpha_i} \cap \tilde{X}_{\alpha_{i+1}} \neq \emptyset$ for every $i \in \{0, 1, \dots, n-1\}$ and $\sum_{i=0}^n z_{|\alpha_i|} < d^\mu(x_1, y_1) + \varepsilon$. There exists $k_\varepsilon \in \mathbb{N}$ such

that $z_{|\alpha_i|} = z_{|\alpha_i|}^{k_\varepsilon}$ for every $i \in \{0, 1, \dots, n\}$. Hence $d^{\mu^k}(x_1, y_1) \stackrel{(2)}{\leq} d^{\mu^{k_\varepsilon}}(x_1, y_1) \leq \sum_{i=0}^n z_{|\alpha_i|} < d^\mu(x_1, y_1) + \varepsilon$, so $0 \leq d^{\mu^k}(x_1, y_1) - d^\mu(x_1, y_1) < \varepsilon$ for every $k \in \mathbb{N}$, $k \geq k_\varepsilon$, i.e. $d^\mu(x_1, y_1) = \lim_{k \rightarrow \infty} d^{\mu^k}(x_1, y_1)$.

Finally $d^\mu(x, y) \stackrel{(3)}{\geq} M - \sum_{k=0}^{\infty} (d^{\mu^k}(x, y) - d^{\mu^{k+1}}(x, y)) \stackrel{(1)}{\geq} M - \sum_{k=0}^{\infty} \frac{M}{2^{k+2}} = \frac{M}{2} > 0$, so $d^\mu(x, y) > 0$.

iii) We have $d^\mu = \inf_{k \in \mathbb{N}} d^{\mu^k} \leq d^{\mu^0} = M$. \square

A bounded and complete metric d on X

Proposition 3.13. *In the above framework, there exists a sequence $(\mu_n)_{n \in \mathbb{N}}$ of decreasing sequences such that the function $\rho : X \times X \rightarrow [0, \infty)$, given by $\rho(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^n} d^{\mu_n}(x, y)$ for every $x, y \in X$, is a bounded metric.*

Proof. Let us consider a fixed $M > 0$. For every $x, y \in A$, $x \neq y$, based on Proposition 3.12, there exists a decreasing sequence $\mu_{x,y} = (z_n^{x,y})_{n \in \mathbb{N}}$ such that $d^{\mu_{x,y}}(x, y) > 0$, $\lim_{n \rightarrow \infty} z_n^{x,y} = 0$ and $d^{\mu_{x,y}} \leq M$. In the sequel, by τ_A we mean the topology on A that was defined on the proof of Theorem 3.4, while by d we mean the metric on A given by the same result.

Claim 1. *In the above framework, there exist $D_x, D_y \in \tau_A$ such that:*

- i) $x \in D_x$ and $y \in D_y$;
- ii) $d^{\mu_{x,y}}(u, v) > 0$ for every $u \in D_x$ and every $v \in D_y$.

Justification of claim 1. Since $\lim_{n \rightarrow \infty} z_n^{x,y} = 0$, we can choose $n \in \mathbb{N}$ such that $z_n^{x,y} < \frac{d^{\mu_{x,y}}(x, y)}{4}$. Then $D_x \stackrel{\text{not}}{=} \bigcup_{|\alpha|=n, x \in A_\alpha} A_\alpha \in \tau_A$ (since, according to the observation made after Claim 1 from the proof of Theorem 3.4, we have $A = \bigcup_{|\alpha|=n} A_\alpha = \bigcup_{|\alpha|=n, x \in A_\alpha} A_\alpha \cup \bigcup_{|\alpha|=n, x \notin A_\alpha} A_\alpha$ and $\bigcup_{|\alpha|=n, x \notin A_\alpha} A_\alpha$ is compact as a finite union of compact sets) and $D_y \stackrel{\text{not}}{=} \bigcup_{|\alpha|=n, y \in A_\alpha} A_\alpha \in \tau_A$ (the same argument). For $u \in D_x$ and $v \in D_y$, since $d^{\mu_{x,y}}(x, u) \leq z_n^{x,y} < \frac{d^{\mu_{x,y}}(x, y)}{4}$ and $d^{\mu_{x,y}}(y, v) \leq z_n^{x,y} < \frac{d^{\mu_{x,y}}(x, y)}{4}$, we have $d^{\mu_{x,y}}(u, v) \geq d^{\mu_{x,y}}(x, y) - d^{\mu_{x,y}}(x, u) - d^{\mu_{x,y}}(y, v) \geq d^{\mu_{x,y}}(x, y) - 2 \frac{d^{\mu_{x,y}}(x, y)}{4} = \frac{d^{\mu_{x,y}}(x, y)}{2} > 0$ and the justification of the claim is done.

Hence, for every $\varepsilon > 0$, from the open cover (provided by Claim 1) $(D_x \times D_y)_{x,y \in A \times A}$ of the compact set $K_\varepsilon \stackrel{\text{not}}{=} \{(x, y) \in A \times A \mid d(x, y) \geq \varepsilon\}$ we can extract a finite open cover, so there exist the decreasing sequences $\mu^1, \dots, \mu^{p_\varepsilon}$ such that for every $(x, y) \in K_\varepsilon$ there exists $j_{x,y} \in \{1, 2, \dots, p_\varepsilon\}$ having the property that $d^{\mu^{j_{x,y}}}(x, y) > 0$. Consequently, as $\{(x, y) \in A \times A \mid x \neq y\} = \bigcup_{n \in \mathbb{N}} K_{\frac{1}{n}}$, there exists a sequence $(\mu_n)_{n \in \mathbb{N}}$ of decreasing sequences such that for every $(x, y) \in A \times A$, $x \neq y$, we can find $n_{x,y} \in \mathbb{N}$ having the property that $d^{\mu_{n_{x,y}}}(x, y) > 0$. Moreover

$$d^{\mu_n} \leq M \tag{1}$$

and

$$\lim_{k \rightarrow \infty} z_k^n = 0, \quad (2)$$

for every $n \in \mathbb{N}$, where $\mu_n = (z_k^n)_{k \in \mathbb{N}}$.

Now we define the function $\rho : X \times X \rightarrow [0, \infty)$ by $\rho(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^n} d^{\mu_n}(x, y)$

for every $x, y \in X$. As $d^{\mu_n} \stackrel{(1)}{\leq} M$ for every $n \in \mathbb{N}$, ρ is well defined and, moreover, $\rho \leq 2M$ for every $x, y \in X$, i.e. ρ is bounded. It is clear that $\rho(x, x) = 0$, $\rho(x, y) = \rho(y, x)$ and $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for every $x, y, z \in X$. Moreover, $\rho(x, y) = 0 \Rightarrow x = y$ for every $x, y \in X$. Indeed, for $x, y \in X$, $x \neq y$, we divide the discussion into the following cases: a) $x, y \in A$; b) $x \in X \setminus A$ or $y \in X \setminus A$. In case a), we can find $n_{x,y} \in \mathbb{N}$ having the property that $d^{\mu_{n_{x,y}}}(x, y) > 0$, so $\rho(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^n} d^{\mu_n}(x, y) \geq \frac{1}{2^{n_{x,y}}} d^{\mu_{n_{x,y}}}(x, y) > 0$, hence $\rho(x, y) > 0$. In situation b), from Proposition 3.8, d) , we infer that $d^{\mu_n}(x, y) > 0$ for every $n \in \mathbb{N}$, so $\rho(x, y) > 0$. We conclude that ρ is a metric. \square

In the above framework, we consider the sequence $\eta = (z_k)_{k \in \mathbb{N}}$, where $z_k = \sum_{n=0}^{\infty} \frac{1}{2^n} z_k^n$.

Proposition 3.14 (The properties of the sequence η). *In the above framework, the sequence η has the following properties:*

- a) *it is well define;*
- b) *it is decreasing;*
- c) $\lim_{k \rightarrow \infty} z_k = 0$.

Proof.

a) As the series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is convergent and $z_k^n \leq M$ for every $k, n \in \mathbb{N}$, the comparison test yields the conclusion.

b) As $z_{k+1}^n \leq z_k^n$ for every $k, n \in \mathbb{N}$, the same comparison test assures us that $z_{k+1} \leq z_k$ for every $k \in \mathbb{N}$.

c) Let us consider an arbitrary $\varepsilon > 0$. Since $\lim_{k \rightarrow \infty} z_k^0 = \lim_{k \rightarrow \infty} \frac{1}{2} z_k^1 = \dots = \lim_{k \rightarrow \infty} \frac{1}{2^{n_\varepsilon-1}} z_k^{n_\varepsilon-1} = 0$, where $n_\varepsilon = 3 + [\log_2 \frac{M}{\varepsilon}]$, there exists $k_\varepsilon \in \mathbb{N}$ such that $0 < z_k^0 < \frac{\varepsilon}{2^{n_\varepsilon}}, 0 < \frac{1}{2} z_k^1 < \frac{\varepsilon}{2^{n_\varepsilon}}, \dots, 0 < \frac{1}{2^{n_\varepsilon-1}} z_k^{n_\varepsilon-1} < \frac{\varepsilon}{2^{n_\varepsilon}}$ for every $k \in \mathbb{N}, k \geq k_\varepsilon$.

Consequently we have $0 \leq z_k = z_k^0 + \frac{1}{2}z_k^1 + \dots + \frac{1}{2^{n_\varepsilon-1}}z_k^{n_\varepsilon-1} + \frac{1}{2^{n_\varepsilon}}z_k^{n_\varepsilon} + \frac{1}{2^{n_\varepsilon+1}}z_k^{n_\varepsilon+1} + \dots + \frac{1}{2^n}z_k^n + \dots \leq n_\varepsilon \frac{\varepsilon}{2^{n_\varepsilon}} + M(\frac{1}{2^{n_\varepsilon}} + \frac{1}{2^{n_\varepsilon+1}} + \dots + \frac{1}{2^n} + \dots) = \frac{\varepsilon}{2} + \frac{M}{2^{n_\varepsilon-1}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for every $k \in \mathbb{N}$, $k \geq k_\varepsilon$ and the conclusion follows. \square

Now we can consider the semi-metric $d^\eta \stackrel{not}{=} \delta$.

Proposition 3.15 (The properties of the metric δ). *In the above framework, δ has the following properties:*

- a) $\rho \leq \delta$;
- b) $\delta \leq 2M$;
- c) (X, δ) is a bounded and complete metric space.

Proof.

a) For $x, y \in X$, $x \neq y$, $p \in \mathbb{N}$ and $\alpha_0, \alpha_1, \dots, \alpha_p \in \Lambda^*(I)$ such that $x \in \tilde{X}_{\alpha_0}$, $y \in \tilde{X}_{\alpha_n}$ and $\tilde{X}_{\alpha_i} \cap \tilde{X}_{\alpha_{i+1}} \neq \emptyset$ for every $i \in \{0, 1, \dots, p-1\}$, we have $\frac{1}{2^n}d^{\mu_n}(x, y) \leq \frac{1}{2^n} \sum_{i=0}^p z_{|\alpha_i|}^n$ for every $n \in \mathbb{N}$, so $\sum_{n=0}^{\infty} \frac{1}{2^n}d^{\mu_n}(x, y) \leq \sum_{n=0}^{\infty} \frac{1}{2^n}z_{|\alpha_0|}^n + \dots + \sum_{n=0}^{\infty} \frac{1}{2^n}z_{|\alpha_p|}^n = z_{|\alpha_0|} + \dots + z_{|\alpha_p|}$. Hence $\rho(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^n}d^{\mu_n}(x, y) \leq d^\eta(x, y) = \delta(x, y)$. As the last inequality is also true for $x = y$, the justification of a) is done.

b) As $z_k^n \leq M$ for every $k, n \in \mathbb{N}$, we deduce that $z_k = \sum_{n=0}^{\infty} \frac{1}{2^n}z_k^n \leq M \sum_{n=0}^{\infty} \frac{1}{2^n} = 2M$ for every $k \in \mathbb{N}$. Hence $\eta \prec \theta$, where $\theta = (y_k)_{k \in \mathbb{N}}$, $y_k = 2M$ for every $k \in \mathbb{N}$, and we infer that $\delta = d^\eta \leq d^\theta = 2M$.

c) Since $\delta(x, y) = 0 \stackrel{a)}{\Rightarrow} \rho(x, y) = 0 \stackrel{\text{Proposition 3.13}}{\Rightarrow} x = y$ we conclude that δ is a metric on X . According to b) it is bounded. In order to prove that (X, δ) is complete, let us consider a Cauchy sequence $(x_n)_{n \in \mathbb{N}}$. By passing to a subsequence, we divide the discussion into the following two cases (see the proof of Proposition 3.8 for the definition of the function m): a) there exists $N \in \mathbb{N}$ such that $m(x_n) \leq N$ for every $n \in \mathbb{N}$; b) $\lim_{n \rightarrow \infty} m(x_n) = \infty$.

In the first case, we have $\delta(x_n, x_m) = \inf \left\{ \sum_{i=0}^p z_{|\alpha_i|} \mid \text{there exist } p \in \mathbb{N} \text{ and } \alpha_0, \alpha_1, \dots, \alpha_p \in \Lambda^*(I) \text{ such that } x_n \in \tilde{X}_{\alpha_0}, x_m \in \tilde{X}_{\alpha_p} \text{ and } \tilde{X}_{\alpha_i} \cap \tilde{X}_{\alpha_{i+1}} \neq \emptyset \right\}$ for every $i \in \{0, 1, \dots, p-1\}$ $\stackrel{(z_n)_{n \in \mathbb{N}} \text{ is decreasing}}{\geq} z_N$ for every $x_n \neq x_m$, so, as $(x_n)_{n \in \mathbb{N}}$ is Cauchy, there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1} = x_{n_0+2} = \dots$

and consequently the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent. In the second case, for each $n \in \mathbb{N}$, there exists $\alpha_n \in \Lambda^*(I)$ such that $x_n \in \tilde{X}_{\alpha_n}$ and $|\alpha_n| = m(x_n)$. Therefore an argument similar to the one used in the proof of Proposition 3.11 assures us that one can pick $\alpha \in \Lambda(I)$ such that $[\alpha]_n = x_n \in \tilde{X}_{[\alpha]_n}$ for every $n \in \mathbb{N}$. Hence $\delta(x_n, a_\alpha) \leq z_{m(x_n)}$ for every $n \in \mathbb{N}$ which implies that the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent (having the limit a_α). \square

Now let us consider a fixed strictly increasing sequence $(c_n)_{n \in \mathbb{N}}$ such that $c_0 = 1$, $(\frac{c_n}{c_{n+1}})_{n \in \mathbb{N}}$ is strictly increasing and $c_n \leq 2$, $\frac{1}{2} \leq \frac{c_n}{c_{n+1}}$ for every $n \in \mathbb{N}$ and the function $d : X \times X \rightarrow [0, \infty)$ given by $d(x, y) = \sup_{\alpha \in \Lambda^*(I)} c_{|\alpha|} \delta(f_\alpha(x), f_\alpha(y))$ for every $x, y \in X$.

Proposition 3.16. *In the above framework, (X, d) is a bounded and complete metric space.*

Proof. It follows from the inequality $\delta \leq d \stackrel{\text{Proposition 3.8, e)}}{\leq} 2\delta$ and the fact that, according to Proposition 3.15, c), (X, δ) is a bounded and complete metric space. \square

A word of warning: Even though we use the same notation, namely d , for the metric from Theorem 3.4 and for the one from Proposition 3.16, it is clear that they are different objects, the first one being a distance on A , while the second one is a metric on X .

A comparison function φ which makes φ -contractions with respect to d all the functions of the family having attractor

Lemma. 3.17. *In the above framework, we have $d(X_\alpha) \leq d(\tilde{X}_\alpha) \leq 2z_{|\alpha|}$ for every $\alpha \in \Lambda^*(I)$.*

Proof. For every $x, y \in \tilde{X}_\alpha$ and $\beta \in \Lambda^*(I)$ we have $\delta(f_\beta(x), f_\beta(y)) \stackrel{\text{Proposition 3.7, e)}}{\leq} \sup_{\gamma \in \tilde{X}_{\beta\alpha}} \delta(f_\gamma(x), f_\gamma(y)) \leq z_{|\beta\alpha|} \leq z_{|\alpha|}$, so $d(x, y) = \sup_{\beta \in \Lambda^*(I)} c_{|\beta|} \delta(f_\beta(x), f_\beta(y)) \leq$

$2z_{|\alpha|}$ and consequently $d(X_\alpha) \leq d(\tilde{X}_\alpha) \leq 2z_{|\alpha|}$ for every $\alpha \in \Lambda^*(I)$. \square

Let us fix $M > 0$. Taking into account Proposition 3.14, c) and Lemma 3.17, for every $r \in (0, 4M]$, there exists $n_r \in \mathbb{N}$ such that $d(X_\alpha) \leq d(\tilde{X}_\alpha) \leq$

$\frac{r}{20}$ for every $\alpha \in \Lambda^*(I)$ with the property that $|\alpha| \geq n_r$. For every $r \in (0, 4M)$ we consider the comparison function $\varphi_r : [0, \infty) \rightarrow [0, \infty)$, given

$$\varphi_r(x) = \begin{cases} 0, & x \in [0, r - \rho_r) \\ \frac{c_{n_r}}{c_{n_r}+1}x, & x \in [r - \rho_r, r + \rho_r] \\ \frac{c_{n_r}}{c_{n_r}+1}(r + \rho_r), & x \in (r + \rho_r, \infty) \end{cases}, \text{ where } \rho_r \in (0, \min\{4M - r, \frac{r}{2}\}).$$

We also consider the comparison function $\varphi_{4M} : [0, \infty) \rightarrow [0, \infty)$,

$$\varphi_{4M}(x) = \begin{cases} 0, & x \in [0, 2M) \\ \frac{c_{n_M}}{c_{n_M}+1}x, & x \in [2M, 4M] \\ \frac{c_{n_M}}{c_{n_M}+1}4M, & x \in (4M, \infty) \end{cases}.$$

Lemma. 3.18. *In the above framework, we have $d(f_i(x), f_i(y)) \leq \varphi_r(d(x, y))$ for every $i \in I$, $r \in (0, 4M)$ and $x, y \in X$ having the property that $d(x, y) \in [r - \rho_r, r + \rho_r]$. Moreover $d(f_i(x), f_i(y)) \leq \varphi_{4M}(d(x, y))$ for every $i \in I$ and $x, y \in X$ having the property that $d(x, y) \in [2M, 4M]$.*

Proof. We treat only the situation $r \in (0, 4M)$ (the proof for $r = 4M$ being similar). We divide the discussion into two cases:

a) $d(f_i(x), f_i(y)) < \frac{r}{10}$;

b) $\frac{r}{10} \leq d(f_i(x), f_i(y))$.

In the first case we have $d(f_i(x), f_i(y)) < \frac{r}{10} < \frac{r}{4} = \frac{1}{2} \frac{r}{2} \leq \frac{1}{2}(r - \rho_r) \leq \frac{1}{2}d(x, y) \leq \frac{c_{n_r}}{c_{n_r}+1}d(x, y) = \varphi_r(d(x, y))$.

In the second case, noting that $\delta(f_\alpha(f_i(x)), f_\alpha(f_i(y))) \leq d(f_\alpha(f_i(x)), f_\alpha(f_i(y))) \leq d(X_\alpha) \leq \frac{r}{20} < \frac{r}{10}$ for every $\alpha \in \Lambda^*(I)$ with the property that $|\alpha| \geq n_r$, we conclude that $d(f_i(x), f_i(y)) = \sup_{\alpha \in \Lambda^*(I)} c_{|\alpha|} \delta(f_\alpha(f_i(x)), f_\alpha(f_i(y))) =$

$\max_{\alpha \in \Lambda^*(I), |\alpha| \leq n_r} c_{|\alpha|} \delta(f_\alpha(f_i(x)), f_\alpha(f_i(y)))$, so there exists $\alpha_0 \in \Lambda^*(I)$ such that $|\alpha_0| \leq n_r$ and

$$d(f_i(x), f_i(y)) = c_{|\alpha_0|} \delta(f_{\alpha_0}(f_i(x)), f_{\alpha_0}(f_i(y))). \quad (1)$$

As $c_{|\alpha_{0i}|} \delta(f_{\alpha_{0i}}(x), f_{\alpha_{0i}}(y)) \leq d(x, y)$, using (1), we get $\frac{c_{|\alpha_{0i}|}}{c_{|\alpha_0|}} d(f_i(x), f_i(y)) \leq d(x, y)$, i.e. $d(f_i(x), f_i(y)) \leq \frac{c_{|\alpha_0|}}{c_{|\alpha_0|}+1} d(x, y) \leq \frac{c_{n_r}}{c_{n_r}+1} d(x, y) = \varphi_r(d(x, y))$. \square

Note that the family consisting of the intervals $(2M, 5M)$ and $(r - \rho_r, r + \rho_r)$, where $r \in (0, 4M)$, is an open cover of $(0, 4M]$ which is Lindelöf and paracompact, so there exists a sequence $(r_n)_{n \in \mathbb{N}}$ of elements from $(0, 4M]$ such that $(0, 4M] = \bigcup_{n \in \mathbb{N}} [r_n - \rho_{r_n}, r_n + \rho_{r_n}]$ and the family $\{[r_n - \rho_{r_n}, r_n + \rho_{r_n}]\}_{n \in \mathbb{N}}$

is locally finite, where by $[r_n - \rho_{r_n}, r_n + \rho_{r_n}]$ we mean $[2M, 4M]$ in case that $r_n = 4M$.

Lemma 3.19. *In the above framework, the function $\varphi = \sup_{n \in \mathbb{N}} \varphi_{r_n}$ is a comparison function.*

Proof. It is obvious that φ is increasing and that $\varphi(t) < t$ for every $t > 0$ since all the functions φ_{r_n} have these properties. Moreover, for every $t \in [0, \infty)$ there exists a neighborhood V_t of t which intersects only a finite number of intervals $[r_n - \rho_{r_n}, r_n + \rho_{r_n}]$ and consequently $\varphi|_{V_t}$ is continuous since it can be presented as the maximum of a finite set of continuous functions. Hence φ is continuous. \square

Lemma 3.20. *In the above framework, all the functions f_i are φ -contractions.*

Proof. For $x, y \in X$, $x \neq y$, we have $d(x, y) \in (0, 4M] = \bigcup_{n \in \mathbb{N}} [r_n - \rho_{r_n}, r_n + \rho_{r_n}]$, so there exists $n_0 \in \mathbb{N}$ such that $d(x, y) \in [r_{n_0} - \rho_{r_{n_0}}, r_{n_0} + \rho_{r_{n_0}}]$ and therefore we have $d(f_i(x), f_i(y)) \stackrel{\text{Lemma 3.18}}{\leq} \varphi_{r_{n_0}}(d(x, y)) \leq \varphi(d(x, y))$ for every $i \in I$. As the last inequality is obviously true for $x = y$, we conclude that f_i is φ -contraction for every $i \in I$. \square

We summarize the above facts in the following:

Theorem 3.21. *Given a family of functions $(f_i)_{i \in I}$ having attractor, there exists a metric d on X and a comparison function φ such that:*

- a) *the metric space (X, d) is complete and bounded;*
- b) *f_i is φ -contraction with respect to d for every $i \in I$.*

Combining Proposition 3.1 and Theorem 3.21 we obtain the following:

Theorem 3.22. *Given $(f_i)_{i \in I}$ a family of functions, where $f_i : X \rightarrow X$ and I is finite, the following two statements are equivalent:*

- I. *There exists a metric d on X and a comparison function φ such that:*
 - a) *the metric space (X, d) is complete and bounded;*
 - b) *f_i is φ -contraction with respect to d for every $i \in I$.*
- II. *The following two statements are valid:*

a) *For every $\alpha \in \Lambda(I)$, the set $\bigcap_{n \in \mathbb{N}} X_{[\alpha]_n}$ has a unique element which is denoted by a_α .*

b) If $a_\alpha \neq a_\beta$, where $\alpha, \beta \in \Lambda(I)$, then there exists $n_0 \in \mathbb{N}$ such that $X_{[\alpha]n_0} \cap X_{[\beta]n_0} = \emptyset$.

4. FINAL REMARKS

The unbounded case

The following result removes the boundedness restriction on the metric d .

Theorem 4.1. *Given $(f_i)_{i \in I}$ a family of functions, where $f_i : X \rightarrow X$ and I is finite, the following two statements are equivalent:*

- I. *There exists a metric D on X and a comparison function φ such that:*
 - a) *the metric space (X, D) is complete;*
 - b) *f_i is φ -contraction with respect to D for every $i \in I$.*
- II. *There exists a subset X_1 of X such that the following four statements are valid:*
 - a) $F(X_1) \subseteq X_1$.
 - b) *For every $\alpha \in \Lambda(I)$, the set $\bigcap_{n \in \mathbb{N}} (X_1)_{[\alpha]n}$ has a unique element which is denoted by a_α .*
 - c) *If $a_\alpha \neq a_\beta$, where $\alpha, \beta \in \Lambda(I)$, then there exists $n_0 \in \mathbb{N}$ such that $(X_1)_{[\alpha]n_0} \cap (X_1)_{[\beta]n_0} = \emptyset$.*
 - d) *For every $x \in X$ there exists $n_x \in \mathbb{N}$ such that $F^{[n_x]}(\{x\}) \subseteq X_1$, where $F : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is given by $F(C) = \bigcup_{i \in I} f_i(C)$ for every subset C of X .*

Proof.

I) \Rightarrow II) We choose $X_1 = B(A, r)$, where $r > 0$ and A is the unique fixed point of the function F_S defined on Remark 3.2, ii). For the verification of a) we choose $v \in F(B(A, r))$. Then there exists $i \in I$ and $x \in B(A, r)$ such that $v = f_i(x)$ and there exists $y \in A$ such that $d(x, y) < r$. Then $d(v, f_i(y)) = d(f_i(x), f_i(y)) \leq \varphi(d(x, y)) < \varphi(r)$ and consequently, as $f_i(y) \in f_i(A) \subseteq A$, we conclude that $v \in B(A, r)$. Hence $F(B(A, r)) \subseteq B(A, r)$. The properties b) and c) can be proved with exactly the same techniques used in the proof of Proposition 3.1. The property d) results from the fact that $\lim_{n \rightarrow \infty} h(F^{[n]}(\{x\}), A) = 0$ (see Remark 3.2, ii)).

II) \Rightarrow I) According to Theorem 3.21, taking into account b) and c), there exists a metric d on X_1 and a comparison function φ such that: a) the metric

space (X_1, d) is complete and bounded; b) f_i is φ -contraction with respect to d for every $i \in I$.

For a given $a \in (0, 1)$, the function $\psi : [0, \infty) \rightarrow [0, \infty)$ given by

$$\psi(t) = \sup_{x \in [0, t]} \{ax + \varphi(t - x)\} = \sup\{\varphi_1(t_1) + \varphi(t_2) \mid t_1, t_2 \geq 0, t_1 + t_2 \leq t\},$$

where $\varphi_1(t) = at$, for every $t \geq 0$, is a comparison function (see Fact 10 from the proof of Theorem 3.1 from [20]).

Note that, taking into account Remark 2.2, ii), we have $\varphi \leq \psi$ and $\varphi_1 \leq \psi$.

We consider the function $D : X \times X \rightarrow [0, \infty)$ given by

$$D(x, y) = \begin{cases} d(x, y), & x, y \in X_1 \\ \max\{Ma^{-l(x)}, Ma^{-l(y)}\}, & \{x, y\} \cap (X \setminus X_1) \neq \emptyset \text{ and } x \neq y, \\ 0, & x = y \in X \setminus X_1 \end{cases}$$

where $l(x) = \begin{cases} -\infty, & x \in X_1 \\ \min\{n \in \mathbb{N} \mid F^{[n]}(\{x\}) \subseteq X_1\}, & x \in X \setminus X_1 \end{cases}$ and M is an upper bound for d . Note that, according to d), $l(x) \in \mathbb{N}$ for every $x \in X \setminus X_1$. We use the convention that $a^\infty = 0$, so $Ma^{-l(x)} = 0$ for $x \in X_1$. One can routinely check that D is a metric on X .

Moreover,

$$D(f_i(x), f_i(y)) \leq \psi(d(x, y)),$$

for every $i \in I$, $x, y \in X$.

Indeed, if $x, y \in X_1$, then $f_i(x), f_i(y) \overset{a)}{\in} X_1$, so $D(f_i(x), f_i(y)) = d(f_i(x), f_i(y)) \leq \varphi(d(x, y)) \leq \psi(d(x, y)) = \psi(D(x, y))$. If $\{x, y\} \cap (X \setminus X_1) \neq \emptyset$ and $x \neq y$, then we divide the discussion into three cases: 1. $l(x) = l(y) = 1$. 2. $l(x) \geq l(y) > 1$. 3. $l(y) \geq l(x) > 1$. In the first case, as $f_i(x) \in F^{[l(x)]}(\{x\}) \subseteq X_1$, $f_i(y) \in F^{[l(y)]}(\{y\}) \subseteq X_1$, we have $D(f_i(x), f_i(y)) = d(f_i(x), f_i(y)) \leq aa^{-1}M = aD(x, y) = \varphi_1(D(x, y)) \leq \psi(D(x, y))$. In the second case, note that $l(f_i(x)) = l(x) - 1$ and $l(f_i(y)) = l(y) - 1$, so $D(f_i(x), f_i(y)) = \max\{Ma^{-l(f_i(x))}, Ma^{-l(f_i(y))}\} = aD(x, y) = \varphi_1(D(x, y)) \leq \psi(D(x, y))$. The third case is similar with the second one. If $x = y \in X \setminus X_1$ the conclusion is clear. \square

Some facts about the topological structure of (X, d^μ)

In the framework of the third section, let us suppose that σ is a distance on X such that there exist $(c_n)_{n \in \mathbb{N}}$ and $(d_n)_{n \in \mathbb{N}}$ having the following properties:

- a) $\lim_{n \rightarrow \infty} c_n = \lim_{n \rightarrow \infty} d_n = 0$;
 - b) $\sigma(x, y) \geq c_{m(x)}$ for every $x, y \in X$, $x \neq y$, with the convention that $c_\infty = 0$ (for the definition of $m(x)$ see the proof of Theorem 3.8, d));
 - c) $d(X_\alpha) \leq d_{|\alpha|}$ for every $\alpha \in \Lambda^*(I)$.
- Let us denote by τ the topology induced by σ .

Then one can easily check the following properties:

- i) the sets \tilde{X}_α are closed with respect to τ ;
- ii) $\{x\}$ is open with respect to τ for every $x \in X \setminus A$;
- iii) $(V_{x,n})_{n \in \mathbb{N}^*}$ is a neighborhood basis for x with respect to τ , where
$$V_{x,n} = \bigcup_{\alpha \in \Lambda^*(I), |\alpha|=n, x \in X_\alpha} \tilde{X}_\alpha \text{ for every } x \in A;$$
- iv) the function $\pi : \Lambda(I) \rightarrow A$, given by $\pi(\alpha) = a_\alpha$ for every $\alpha \in \Lambda(I)$, is continuous with respect to τ ;
- v) If $(x_n)_{n \in \mathbb{N}}$ is a sequence of elements from X and $x \in X$, then:
 - j) for $x \in X \setminus A$: $\lim_{n \rightarrow \infty} x_n = x$ with respect to τ if and only if there exists $n_0 \in \mathbb{N}$ such that $x_n = x$ for every $n \in \mathbb{N}$, $n \geq n_0$;
 - jj) for $x \in A$: $\lim_{n \rightarrow \infty} x_n = x$ with respect to τ if and only if for every $m \in \mathbb{N}$ there exists $n_m \in \mathbb{N}$ having the property that for every $n \in \mathbb{N}$, $n \geq n_m$ there exists $\alpha^n \in \Lambda(I)$ such that $x = a_{\alpha^n}$ and $x_n \in X_{[\alpha^n]_m}$ for every $n \in \mathbb{N}$;
- vi) (X, σ) is complete.

Note that if d^μ is a distance, where $\mu = (\alpha^n)_{n \in \mathbb{N}}$ for some $\alpha \in (0, 1)$, satisfies the requirements imposed on the metric σ from the previous paragraph. Indeed, take $c_n = d_n = \alpha^n$ for every $n \in \mathbb{N}$ and note that a) is obvious, b) results from the proof of Theorem 3.8 and c) could be obtained directly from the definition of d^μ . Consequently, according to vi), (X, d^μ) is complete.

The particular case of a family consisting of one function

For the particular case of a family $(f_i)_{i \in I}$ having the property that the set I has one element, we obtain the following converse of Browder's theorem:

Proposition 4.2. *Given a set X and a function $f : X \rightarrow X$ such that $\bigcap_{n \in \mathbb{N}} f^{[n]}(X)$ is a singleton, there exist a bounded and complete metric d on X and a comparison function φ such that $d(f(x), f(y)) \leq \varphi(d(x, y))$ for every $x, y \in X$.*

Proof. $\mathcal{F} = \{f\}$ is a family of functions having attractor since the second condition from the definition of such a system is obviously valid as $\Lambda(I)$ has just one element and therefore the attractor has just one element. Then just apply Theorem 3.21. \square

Moreover, the following stronger results is valid (see Theorem 5 from [10]):

Proposition 4.3. *Given a set X , $\alpha \in (0, 1)$ and a function $f : X \rightarrow X$ such that $\bigcap_{n \in \mathbb{N}} f^{[n]}(X)$ is a singleton, there exists a complete and bounded metric d on X such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for every $x, y \in X$.*

Proof. $\mathcal{F} = \{f\}$ is a family of functions having attractor consisting of just one element. Hence, given $\alpha \in (0, 1)$, Proposition 3.8, f) and g), assures us that d^μ is a bounded distance and the same line of arguments used in the proof of Proposition 3.8, e), confirms that $d^\mu(f(x), f(y)) \leq \alpha d^\mu(x, y)$ for every $x, y \in X$, where $\mu = (\alpha^n)_{n \in \mathbb{N}}$. Moreover, according to the note from the end of the previous section, (X, d^μ) is complete. Now just take $\delta = d^\mu$. \square

Note that the condition that $\bigcap_{n \in \mathbb{N}} f^{[n]}(X)$ is a singleton (i.e. there exists a unique $x_0 \in X$ such that $\bigcap_{n \in \mathbb{N}} f^{[n]}(X) = \{x_0\}$) implies that x_0 is the unique fixed point of $f^{[k]}$ for every $k \in \mathbb{N}$.

Indeed, if $\bigcap_{n \in \mathbb{N}} f^{[n]}(X) = \{x_0\}$, then $f^{[k]}(x_0) \in \bigcap_{n \in \mathbb{N}} f^{[n]}(X)$, so $f^{[k]}(x_0) = x_0$, i.e. x_0 is a fixed point of $f^{[k]}$. Moreover, if $x_1 \in X$ is a fixed point of $f^{[k]}$, then $x_1 \in \bigcap_{n \in \mathbb{N}} f^{[n]}(X) = \{x_0\}$, so $x_1 = x_0$ and consequently x_0 is the unique fixed point of $f^{[k]}$.

The particular case of a family of functions having attractor with common fixed point

Note that each of the functions of a family of functions having attractor has a unique fixed point.

Indeed, let us consider $(f_i)_{i \in I}$ a family of functions having attractor. Then, according to the property a) from the definition of a family of functions having attractor, $\bigcap_{n \in \mathbb{N}} f_i^{[n]}(X) = \bigcap_{n \in \mathbb{N}} X_{[\theta]_n}$ is a singleton and if $\bigcap_{n \in \mathbb{N}} f_i^{[n]}(X) = \{x_i\}$, then x_i is the unique fixed point of f_i for every $i \in I$. Here θ is the element of $\Lambda(I)$ having all letters equal to i .

The following proposition is a companion of the result due to Wong (see [24]) that extends Bessaga's theorem for a finite family of commuting functions with common unique fixed point. Note that the commutativity of the family's functions is not part of the hypotheses of our result.

Proposition 4.4. *Given a set X , $\alpha \in (0, 1)$ and a family of functions $(f_i)_{i \in I}$ having attractor, there exists a complete and bounded metric d on X such that $d(f_i(x), f_i(y)) \leq \alpha d(x, y)$ for every $x, y \in X$ and every $i \in I$, provided that there exists $x_0 \in X$ such that $f_i(x_0) = x_0$.*

Proof. We have $x_0 = f_{[\beta]_n}(x_0) = f_{[\gamma]_n}(x_0) \in X_{[\beta]_n} \cap X_{[\gamma]_n}$, so $X_{[\beta]_n} \cap X_{[\gamma]_n} \neq \emptyset$ for every $n \in \mathbb{N}$ and every $\beta, \gamma \in \Lambda(I)$. Based on the conditions from the definition of a family of functions having attractor, we infer that the attractor of $(f_i)_{i \in I}$ has just one element and the same arguments used in the proof of Proposition 4.3 assure us that for the complete and bounded metric $d = d^\mu$, where $\mu = (\alpha^n)_{n \in \mathbb{N}}$, we have $d(f_i(x), f_i(y)) \leq \alpha d(x, y)$ for every $x, y \in X$ and every $i \in I$. \square

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